

# A CONSTRUCTIVE APPROACH TO STATIONARY SCATTERING THEORY

N. A. AZAMOV

**ABSTRACT.** In this paper we give a new and constructive approach to stationary scattering theory for pairs of self-adjoint operators  $H_0$  and  $H_1$  on a Hilbert space  $\mathcal{H}$  which satisfy the following conditions: (i) for any open bounded subset  $\Delta$  of  $\mathbb{R}$ , the operators  $FE_{\Delta}^{H_0}$  and  $FE_{\Delta}^{H_1}$  are Hilbert-Schmidt and (ii)  $V = H_1 - H_0$  is bounded and admits decomposition  $V = F^*JF$ , where  $F$  is a bounded operator with trivial kernel from  $\mathcal{H}$  to another Hilbert space  $\mathcal{K}$  and  $J$  is a bounded self-adjoint operator on  $\mathcal{K}$ . An example of a pair of operators which satisfy these conditions is the Schrödinger operator  $H_0 = -\Delta + V_0$  acting on  $L^2(\mathbb{R}^\nu)$ , where  $V_0$  is a potential of class  $K_\nu$  (see B. Simon, *Schrödinger semigroups*, Bull. AMS **7** (1982), 447–526) and  $H_1 = H_0 + V_1$ , where  $V_1 \in L^\infty(\mathbb{R}^\nu) \cap L^1(\mathbb{R}^\nu)$ . Among results of this paper is a new proof of existence and completeness of wave operators  $W_{\pm}(H_1, H_0)$  and a new constructive proof of stationary formula for the scattering matrix. This approach to scattering theory is based on explicit diagonalization of a self-adjoint operator  $H$  on a sheaf of Hilbert spaces  $\mathcal{S}(H, F)$  associated with the pair  $(H, F)$  and with subsequent construction and study of properties of wave matrices  $w_{\pm}(\lambda; H_1, H_0)$  acting between fibers  $\mathfrak{h}_{\lambda}(H_0, F)$  and  $\mathfrak{h}_{\lambda}(H_1, F)$  of sheaves  $\mathcal{S}(H_0, F)$  and  $\mathcal{S}(H_1, F)$  respectively. The wave operators  $W_{\pm}(H_1, H_0)$  are then defined as direct integrals of wave matrices and are proved to coincide with classical time-dependent definition of wave operators.

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2000 *Mathematics Subject Classification.* 47A55.

*Key words and phrases.* Wave matrix, scattering matrix, wave operator, scattering operator, sheaf of Hilbert spaces, rigged Hilbert space, Schrödinger operator.

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## 1. INTRODUCTION AND PRELIMINARIES

In this paper we develop a new approach to stationary scattering theory of abstract Schrödinger equation  $\psi_t = -iH\psi$ . It is known that in some cases wave operators (see e.g. [BW, RS<sub>2</sub>, T, Y])

$$(1) \quad W_\pm(H_1, H_0) = s - \lim_{t \rightarrow \pm\infty} e^{itH_1} e^{-itH_0} P^{(a)}(H_0)$$

of two self-adjoint operators  $H_0$  and  $H_1$  can be defined in terms of eigenfunction expansions of absolutely continuous parts of both the initial  $H_0$  and perturbed  $H_1$  operators, see e.g. formula (83) in [RS<sub>2</sub>]. This relation between wave operators and eigenfunction expansions was used for instance by A. Ya. Povzner [Po, Po<sub>2</sub>] and T. Ikebe [I] (see also e.g. [Th, ASch, Ag, Ku<sub>2</sub>, Ku<sub>3</sub>, Ku] and [RS<sub>2</sub>, §XI.6]) in the case of Schrödinger operators  $-\Delta + V$  on  $L^2(\mathbb{R}^\nu)$ . The approach of this paper to scattering theory as well as that of [Az] combines the auxiliary space method (see for example [Ag], [RS<sub>2</sub>, §XI.6 Appendix]) and definition of the wave matrix via eigenfunction expansions. In this regard, this method has something in common with potential scattering theory. On the other hand, the premise of the approach presented in this paper is of abstract trace-class type and as such it belongs

to the trace-class method of Birman-Kato theory, as opposed to the smooth method of Kato [Ka, Ka<sub>2</sub>] which originated as an abstraction of potential scattering theory methods (see also [Y, Chapter 4]).

Apart from wave operators (1), in scattering theory there are three other objects of prime importance, the scattering operator

$$\mathbf{S}(H_1, H_0) = W_+^*(H_1, H_0)W_-(H_1, H_0),$$

the wave matrices  $w_{\pm}(\lambda; H_1, H_0)$  and the scattering matrix  $S(\lambda; H_1, H_0)$ , which are related by formulas

$$(2) \quad \mathbf{S}(H_1, H_0) = \int_{\hat{\sigma}(H_0)}^{\oplus} S(\lambda; H_1, H_0) d\rho(\lambda), \quad W_{\pm}(H_1, H_0) = \int_{\hat{\sigma}(H_0)}^{\oplus} w_{\pm}(\lambda; H_1, H_0) d\rho(\lambda),$$

where the wave matrices  $w_{\pm}(\lambda; H_1, H_0)$  are unitary operators  $\mathfrak{h}_{\lambda}(H_0) \rightarrow \mathfrak{h}_{\lambda}(H_1)$  and the scattering matrix is a unitary operator  $\mathfrak{h}_{\lambda}(H_0) \rightarrow \mathfrak{h}_{\lambda}(H_0)$  for a.e.  $\lambda \in \hat{\sigma}(H_0)$ ,  $\hat{\sigma}(H_j)$  is a core of spectrum of  $H_j$  and  $\mathfrak{h}_{\lambda}(H_j), j = 0, 1$ , are fiber Hilbert spaces from the direct integrals

$$(3) \quad \mathcal{F}_j: \mathcal{H}^{(a)}(H_j) \cong \int_{\hat{\sigma}(H_0)}^{\oplus} \mathfrak{h}_{\lambda}(H_j) d\rho(\lambda),$$

which diagonalize absolutely continuous parts of the operators  $H_j, j = 0, 1$ , see for instance [Y]. For the scattering matrix  $S(\lambda; H_1, H_0)$  there exist explicit stationary formulas, which are important in physics. An advantage of the new approach to scattering theory of relatively trace-class perturbations considered in this paper is that in this theory we first construct the operators  $w_{\pm}(\lambda; H_1, H_0)$  on a pre-defined and explicitly described core of the absolutely continuous spectrum. This circumstance has its advantages, since, for instance, in some questions it is necessary to consider a family of scattering matrices  $\{w_{\pm}(\lambda; H_r, H_0): r \in [a, b]\}$  for a.e.  $\lambda \in \mathbb{R}$ . This is impossible unless at the very least for each  $r \in [a, b]$  we know an explicit description (that is, not up to a null set of uncertain nature) of the set of values of  $\lambda$  for which the wave matrices  $w_{\pm}(\lambda; H_r, H_0)$  exist. In potential scattering theory for Schrödinger operators such an explicit description of the core exists, namely it is the set  $(0, \infty) \setminus e_+(H)$ , where  $e_+(H)$  is the discrete set of eigenvalues of the Schrödinger operator  $H = -\Delta + V$ , see e.g. [Ag]. In conventional trace-class method such a description does not exist, since this method relies on abstract spectral theorem for arbitrary self-adjoint operators. None of the numerous versions of the spectral theorem (see for instance [RS]) gives an explicit description of a core of spectrum which can be used for this purpose; moreover it is known to be generally impossible. Unlike the usual trace-class method in scattering theory, our approach addresses this issue by introducing an additional structure into the Hilbert space  $\mathcal{H}$ , on which operators  $H_0$  and  $H_1$  act, in the form of a fixed bounded operator  $F$  with trivial kernel from  $\mathcal{H}$  to possibly another Hilbert space  $\mathcal{K}$ . This additional structure which we call rigging generates the auxiliary Hilbert spaces  $\mathcal{H}_{\pm 1}$  which serve as analogous of the weighted  $L^2$ -spaces  $L^{2,s}(\mathbb{R}^{\nu})$  used in potential scattering (see e.g. [Ag], [RS<sub>2</sub>, §XI.6, Appendix]) and, more importantly, it allows to construct an explicit diagonalization of absolutely continuous parts of both the initial  $H_0$  and perturbed  $H_1$  self-adjoint operators. It may be worth mentioning that the diagonalization process works equally well for any self-adjoint operator which is compatible in a certain

sense with the rigging  $F$ , unlike the situation in the potential scattering theory where the initial operator  $H_0 = -\Delta$  is trivially diagonalized by the Fourier transform, while perturbed operator  $-\Delta + V$  is very hard to diagonalize. Once both operators  $H_0$  and  $H_1$  are diagonalized, the wave matrices can be defined and their properties studied. The core of spectrum for *all* elements  $\lambda$  of which the wave matrices are constructed allows an explicit description, — namely, it is the set of full Lebesgue measure for which abstract limiting absorption principle holds for both  $H_0$  and  $H_1$ . Finally, we note that the new approach allows to avoid many technical difficulties of the trace-class method approach to stationary scattering theory, as it is given for example in [Y], and therefore it is essentially simpler; moreover, some tools used in this approach such as an explicit diagonalization of a self-adjoint operator are of interest on their own. Other objects of scattering theory such as wave operators, the scattering matrix and the scattering operator are defined via the wave matrices. Trace-class version of this approach has found applications to the theory of spectral shift function, cf. [Az].

**1.1. Hilbert-Schmidt rigging.** In case of Hilbert-Schmidt rigging  $F$  this program was carried out in [Az]. Since the perturbation operator  $V = H_1 - H_0$  is assumed to admit decomposition  $V = F^* J F$  with bounded operator  $J$  on  $\mathcal{K}$ , the setting of [Az] covers only trace-class perturbations  $V$ . In this paper we show that the condition “ $F$  is Hilbert-Schmidt” can be replaced by the condition “for any bounded Borel sets  $\Delta$  of real numbers the operators  $F E_\Delta^{H_0}, F E_\Delta^{H_1}$  are Hilbert-Schmidt”. Below we give a brief description of results of [Az], both for the purpose of introduction and as a preliminary material used in this paper.

Assume that we are given a Hilbert space  $\mathcal{H}$  rigged with a Hilbert-Schmidt operator  $F$  with trivial kernel and dense range. Let

$$F = \sum_{j=1}^{\infty} \kappa_j \langle \varphi_j, \cdot \rangle \psi_j, \quad F: \mathcal{H} \rightarrow \mathcal{K},$$

be a fixed Schmidt representation of  $F$ , where  $(\kappa_j)$  are  $s$ -numbers of  $F$ ,  $(\varphi_j)$  is an orthonormal basis of  $\mathcal{H}$ ,  $(\psi_j)$  is an orthonormal basis of  $\mathcal{K}$  (see e.g. [RS]). For any self-adjoint operator  $H$  which acts on such a rigged Hilbert space  $(\mathcal{H}, F)$  we introduce the following objects which depend only on the pair  $(H, F)$ .

I. A set  $\Lambda(H, F)$  of real numbers, defined as follows:  $\lambda \in \Lambda(H, F)$  if and only if (i) the trace-class operator

$$T_{\lambda+iy}(H) := F R_{\lambda+iy}(H) F^* := F (H - \lambda - iy)^{-1} F^*$$

has a limit, denoted by  $T_{\lambda+i0}(H)$ , in uniform operator norm as  $y \rightarrow 0^+$  and (ii) the imaginary part

$$\operatorname{Im} T_{\lambda+iy}(H) = \frac{1}{2i} (T_{\lambda+iy}(H) - T_{\lambda-iy}(H))$$

of  $T_{\lambda+iy}(H)$  has a limit, denoted by  $\operatorname{Im} T_{\lambda+i0}(H)$ , in trace-class norm.

**Theorem 1.1.** *(The abstract limiting absorption principle [BE], see also [Y, §6.1]) The set  $\Lambda(H, F)$  has full Lebesgue measure, that is, the Lebesgue measure of the set  $\mathbb{R} \setminus \Lambda(H, F)$  is zero.*

In fact, the abstract limiting absorption principle asserts that the limit  $T_{\lambda+i0}(H)$  exists for a.e.  $\lambda$  in stronger Hilbert-Schmidt norm, but we shall not need this. We prefer to use the qualifier “abstract” in the name of this limiting absorption principle to distinguish it from the limiting absorption principle for differential operators, see e.g. [Ag, Ku]. The set of full Lebesgue measure  $\Lambda(H, F)$  will play the role of the core of spectrum  $\hat{\sigma}(H)$  mentioned previously. Of course, the set  $\Lambda(H, F)$  can be much larger than the spectrum of  $H$ , but what is important for us is the following property of  $\Lambda(H, F)$ .

**Theorem 1.2.** *The operator  $HE_{\Lambda(H, F)}^H$  is absolutely continuous.*

This theorem asserts that the set  $\Lambda(H, F)$  cuts away from the set of all real numbers the singular part of the spectrum of  $H$ . In particular, the null set  $\mathbb{R} \setminus \Lambda(H, F)$  contains all eigenvalues of  $H$ . Proof can be found in e.g. [Y], see also [Az, Corollary 2.5.3].

II. Now for each  $\lambda \in \Lambda(H, F)$  we introduce the fiber Hilbert space  $\mathfrak{h}_\lambda(H, F)$  as a certain (closed linear) subspace of the Hilbert space  $\ell_2$ . For any  $\lambda \in \Lambda(H, F)$  and  $y > 0$  let (see [Az, (2.10)])

$$(4) \quad \varphi(\lambda + iy) = \frac{1}{\pi} (\langle \psi_i, \operatorname{Im} T_{\lambda+iy}(H) \psi_j \rangle)_{i,j=1}^\infty;$$

this is the matrix of the operator  $\frac{1}{\pi} \operatorname{Im} T_{\lambda+iy}(H)$  in the basis  $(\psi_j)$  of  $\mathcal{K}$ , and therefore  $\varphi(\lambda + iy)$  is a positive trace-class operator on  $\ell_2$ . Let also

$$(5) \quad \eta(\lambda + iy) = \sqrt{\varphi(\lambda + iy)};$$

this is a positive Hilbert-Schmidt operator on  $\ell_2$ . By Theorem 1.1, there exist the trace-class limit  $\varphi(\lambda + i0)$  and the compact norm limit  $\eta(\lambda + i0)$ . We define the fiber Hilbert space  $\mathfrak{h}_\lambda = \mathfrak{h}_\lambda(H, F)$  as a subspace of  $\ell_2$  by

$$(6) \quad \mathfrak{h}_\lambda = \overline{\operatorname{ran} \eta(\lambda + i0)}.$$

Dimension of  $\mathfrak{h}_\lambda$  can be zero too. If necessary we indicate depends of  $\mathfrak{h}_\lambda$  on  $H$ , but we shall always omit  $F$  and write  $\mathfrak{h}_\lambda(H)$ .

III. With a Hilbert-Schmidt rigging  $F$  one can associate in a standard way two Hilbert spaces  $\mathcal{H}_{\pm 1} = \mathcal{H}_{\pm 1}(F)$  with natural Hilbert-Schmidt inclusions (see e.g. [RS<sub>2</sub>, XI.6 Appendix], [Az, §2.6])

$$(7) \quad \mathcal{H}_1(F) \subset \mathcal{H} \subset \mathcal{H}_{-1}(F).$$

The Hilbert space  $\mathcal{H}_1$  is a vector space  $\operatorname{ran}(F^*)$  endowed with scalar product  $\langle \cdot, \cdot \rangle_{\mathcal{H}_1}$  defined by formula

$$\langle F^* \psi', F^* \psi'' \rangle_{\mathcal{H}_1} = \langle \psi', \psi'' \rangle_{\mathcal{K}}.$$

The Hilbert space  $\mathcal{H}_{-1}$  is the closure of  $\mathcal{H}$  endowed with scalar product  $\langle \cdot, \cdot \rangle_{\mathcal{H}_{-1}}$  defined by formula

$$\langle \varphi', \varphi'' \rangle_{\mathcal{H}_{-1}} = \langle F \varphi', F \varphi'' \rangle_{\mathcal{K}}.$$

There exists a natural pairing  $\langle \cdot, \cdot \rangle_{1,-1} : \mathcal{H}_1 \times \mathcal{H}_{-1} \rightarrow \mathbb{C}$  defined by formula: for all  $\varphi', \varphi'' \in \operatorname{ran}(F^*)$

$$\langle \varphi', \varphi'' \rangle_{1,-1} = \langle \varphi', \varphi'' \rangle.$$

IV. For all  $\lambda \in \Lambda(H, F)$  we define the evaluation operator

$$(8) \quad \mathcal{E}_\lambda = \mathcal{E}_\lambda(H, F): \mathcal{H}_1(F) \rightarrow \mathfrak{h}_\lambda$$

as follows. Any vector  $f \in \mathcal{H}_1(F)$  considered as an element of  $\mathcal{H}$  can uniquely be written as

$$f = \sum_{j=1}^{\infty} \beta_j \kappa_j \varphi_j,$$

where  $(\beta_j) \in \ell_2$ . The value of the operator  $\mathcal{E}_\lambda$  at  $f \in \mathcal{H}_1(F)$  is defined by formula

$$(9) \quad \mathcal{E}_\lambda(f) = \sum_{j=1}^{\infty} \beta_j \eta_j(\lambda),$$

where  $\eta_j(\lambda)$  is the  $j$ -th column of the matrix  $\eta(\lambda + i0)$  (see (5)). One can show that this correctly defines a Hilbert-Schmidt operator (8), see [Az, §3.1] for details.

V. We define a direct integral Hilbert space  $\mathcal{H} = \mathcal{H}(H, F)$  by formula

$$\mathcal{H} = \int_{\Lambda(H, F)}^{\oplus} \mathfrak{h}_\lambda(H) d\lambda.$$

As a measurability base of this direct integral one can take functions  $\Lambda(H, F) \ni \lambda \mapsto \mathcal{E}_\lambda(\varphi_j) \in \mathfrak{h}_\lambda$ ,  $j = 1, 2, \dots$ . Recall that it is possible that  $\dim \mathfrak{h}_\lambda = 0$ ; thus, the set  $\Lambda(H, F)$  can be replaced by the set  $\{\lambda \in \Lambda(H, F): \dim \mathfrak{h}_\lambda = 0\}$ , which is a core of absolutely continuous spectrum of  $H$ ; but we prefer to work with the set  $\Lambda(H, F)$ . Further, the choice of Lebesgue measure  $d\lambda$  in definition of  $\mathcal{H}$  is not necessary, but quite natural as we shall see.

VI. One can now consider the operator  $\mathcal{E} = \mathcal{E}(H, F): \mathcal{H} \rightarrow \mathcal{H}$  defined for  $f \in \mathcal{H}_1$  by formula

$$[\mathcal{E}(f)](\lambda) = \mathcal{E}_\lambda(f).$$

**Theorem 1.3.** [Az, Proposition 3.2.1, Proposition 3.3.5, Theorem 3.4.2] *The operator  $\mathcal{E}: \mathcal{H} \rightarrow \mathcal{H}$  is bounded, it vanishes on the singular subspace  $\mathcal{H}^{(s)}(H)$  of  $H$ , it is isometric on the absolutely continuous subspace  $\mathcal{H}^{(a)}(H)$  of  $H$ , and it is onto, that is,  $\text{ran}(\mathcal{E}) = \mathcal{H}$ . Further, the operator  $\mathcal{E}$  diagonalizes the absolutely continuous part of  $H$ , that is, for all  $f \in \text{dom}(H)$  and for a.e.  $\lambda \in \Lambda(H, F)$ , we have*

$$(10) \quad [\mathcal{E}(Hf)](\lambda) = \lambda \mathcal{E}_\lambda(f).$$

This theorem shows that the operator  $\mathcal{E}$  is a version of the operator  $\mathcal{F}_j$  from (3). But unlike (3), the set  $\Lambda(H, F)$ , the family of Hilbert spaces  $\{\mathfrak{h}_\lambda: \lambda \in \Lambda(H, F)\}$  and the operator  $\mathcal{E}$  are explicitly constructed without a.e. ambiguity. Further, if  $f \in \mathcal{H}_1$  then the right hand side of (10) is defined for all  $\lambda \in \Lambda(H, F)$ .

VII. Now assume that we are given two self-adjoint operators  $H_0$  and  $H_1$  on a rigged Hilbert space  $(\mathcal{H}, F)$ , such that the perturbation  $V = H_1 - H_0$  admits decomposition  $F^* J F$ , where  $J$  is a bounded self-adjoint operator on  $\mathcal{K}$ . Since on one hand  $F$  can be treated as an isomorphism of Hilbert spaces  $\mathcal{H}_{-1}$  and  $\mathcal{K}$  and on the other hand  $F^*$  can be treated as isomorphism of  $\mathcal{K}$  and  $\mathcal{H}_1$ , it follows that the operator  $V = F^* J F$  can be

treated as a bounded operator  $V: \mathcal{H}_{-1} \rightarrow \mathcal{H}_1$ . The operator  $V$  considered as acting from  $\mathcal{H}$  to  $\mathcal{H}$  is a composition (in appropriate order) of the bounded operator  $V: \mathcal{H}_{-1} \rightarrow \mathcal{H}_1$  with two Hilbert-Schmidt inclusions (7). Further, Theorem 1.1 shows that for every  $\lambda \in \Lambda(H_0, F) \cap \Lambda(H_1, F)$  the limits  $R_{\lambda+i0}(H_j)$ ,  $j = 0, 1$ , exist in uniform operator norm, if the operators  $R_{\lambda+iy}(H_j)$  are considered as follows

$$R_{\lambda+iy}(H_j): \mathcal{H}_1(F) \rightarrow \mathcal{H}_{-1}(F).$$

Similarly, the operators  $\text{Im } R_{\lambda+i0}(H_j)$  can be treated as a trace-class operator  $\mathcal{H}_1(F) \rightarrow \mathcal{H}_{-1}(F)$ . Therefore, we can define a trace class operator

$$\mathbf{a}_{\pm}(\lambda; H_1, H_0): \mathcal{H}_1(F) \rightarrow \mathcal{H}_{-1}(F)$$

by formula (compare with [Y, (2.7.4)])

$$\mathbf{a}_{\pm}(\lambda; H_1, H_0) = [1 - R_{\lambda+i0}(H_1)V] \cdot \frac{1}{\pi} \text{Im } R_{\lambda+i0}(H_0).$$

**Theorem 1.4.** [Az, §5.3] *For all  $\lambda \in \Lambda(H_0, F) \cap \Lambda(H_1, F)$  there exists a unique (for each sign) bounded operator*

$$(11) \quad w_{\pm}(\lambda; H_1, H_0): \mathfrak{h}_{\lambda}(H_0) \rightarrow \mathfrak{h}_{\lambda}(H_1),$$

*such that for all  $f, g \in \mathcal{H}_1$  there holds the equality*

$$(12) \quad \langle \mathcal{E}_{\lambda}(H_1)f, w_{\pm}(\lambda; H_1, H_0)\mathcal{E}_{\lambda}(H_0)g \rangle = \langle f, \mathbf{a}_{\pm}(\lambda; H_1, H_0)g \rangle_{1,-1}.$$

*The operators (11), thus defined, have the following properties:*

- (a) *the operators (11) are unitary.*
- (b)  *$w_{\pm}(\lambda; H_0, H_0) = 1_{\mathfrak{h}_{\lambda}}$  and  $w_{\pm}^*(\lambda; H_1, H_0) = w_{\pm}(\lambda; H_0, H_1)$ .*
- (c) *for any three self-adjoint operators  $H_0, H_1$  and  $H_2$  such that  $H_2 - H_1$  and  $H_1 - H_0$  admit decompositions  $F^*J_1F$  and  $F^*J_0F$  with bounded  $J_0, J_1: \mathcal{K} \rightarrow \mathcal{K}$  there hold the equalities*

$$w_{\pm}(\lambda; H_2, H_0) = w_{\pm}(\lambda; H_2, H_1)w_{\pm}(\lambda; H_1, H_0).$$

Once the operators  $w_{\pm}(\lambda; H_1, H_0)$  have been constructed, one defines the wave operators  $W_{\pm}(H_1, H_0)$  and the scattering operator  $\mathbf{S}(H_1, H_0)$  by formulas (2), where the scattering matrix  $S(\lambda; H_1, H_0)$  is defined as an operator  $w_{+}^*(\lambda; H_1, H_0)w_{-}(\lambda; H_1, H_0)$ . It is shown in [Az] that thus defined objects of scattering theory possess well-known properties such as multiplicative property, the stationary formula and agreement with classical time-dependent definitions.

**1.2. Non-compact rigging.** It turns out that the method discussed above can be adjusted for pairs  $H, F$  of operators such that  $FE_{\Delta}$  is Hilbert-Schmidt, where  $E_{\Delta}$  is a spectral projection of  $H$  and  $\Delta$  is a bounded measurable set. Here we describe briefly the main idea. Let  $F: \mathcal{H} \rightarrow \mathcal{K}$  be a bounded operator with trivial kernel and co-kernel, considered as a rigging in Hilbert space  $\mathcal{H}$ , and let  $H$  be a self-adjoint operator on  $\mathcal{H}$ , such that

$$(13) \quad FE_{\Delta}^H \text{ is Hilbert-Schmidt}$$

for all bounded open intervals  $\Delta$ . Since  $FE_\Delta^H$  is Hilbert-Schmidt, we define the set  $\Lambda(H, F)$  as

$$\Lambda(H, F) = \bigcup_{\Delta} (\Lambda(HE_\Delta, FE_\Delta) \cap \Delta),$$

where the union is taken over all bounded open sets  $\Delta \subset \mathbb{R}$ . If  $\lambda \in \Lambda(H, F) \cap \Delta$  then for the operator  $HE_\Delta$  on the Hilbert space  $E_\Delta \mathcal{H}$  one can construct the fiber Hilbert spaces  $\mathfrak{h}_\lambda(\Delta)$ , the evaluation operator  $\mathcal{E}_\lambda^\Delta$ , etc, using the operator  $FE_\Delta: E_\Delta \mathcal{H} \rightarrow \mathcal{K}$  as a Hilbert-Schmidt rigging. A difficulty here is that these objects depend on a choice of an interval  $\Delta$ . It is shown that the Hilbert spaces  $\mathfrak{h}_\lambda(\Delta)$  for different bounded open sets  $\Delta$ , containing  $\lambda$ , are naturally isomorphic and the evaluation operators  $\mathcal{E}_\lambda^\Delta(H)$  can be naturally identified via the unitary operators used to identify the Hilbert spaces  $\mathfrak{h}_\lambda(\Delta)$ . The collection of Hilbert spaces

$$\mathfrak{h}_\lambda(H, F) = \{\mathfrak{h}_\lambda(\Delta): \Delta \text{ is bounded, open and } \lambda \in \Delta\}$$

form a sheaf of fiber Hilbert spaces and the collection of operators

$$\mathcal{E}_\lambda(H) = \{\mathcal{E}_\lambda^\Delta(H): \Delta \text{ is bounded, open and } \lambda \in \Delta\}$$

can be considered as an operator

$$\mathcal{E}_\lambda(H): \mathcal{H}_1(F) \rightarrow \mathfrak{h}_\lambda(H, F),$$

which is a generalization of the evaluation operator (8), where  $\mathcal{H}_{\pm 1}(F)$  are Hilbert spaces generated by the rigging  $F$ . It is shown (Theorem 1.3) that extension of the operator  $\mathcal{E}_\lambda(H)$  to the Hilbert space associated with the direct integral

$$\mathcal{S}(H, F) := \int_{\Lambda(H, F)}^{\oplus} \mathfrak{h}_\lambda(H, F) d\lambda$$

diagonalizes the operator  $H$ . Once this is done, the rest of the theory is constructed similarly to the case of Hilbert-Schmidt rigging  $F$ . For example, the stationary formula (Theorem 5.2) for the scattering matrix takes the form

$$S(\lambda; H_1, H_0) = 1_{\mathfrak{h}_\lambda} - 2\pi i \mathcal{E}_\lambda(H_0) V (1 + R_{\lambda+i0}(H_0) V)^{-1} \mathcal{E}_\lambda^\diamond(H_0),$$

where the operators in the right hand side are understood as follows

$$\mathfrak{h}_\lambda(H, F) \xleftarrow{\mathcal{E}_\lambda(H_0)} \mathcal{H}_1(F) \xleftarrow{V} \mathcal{H}_{-1}(F) \xleftarrow{R_{\lambda+i0}(H_0)} \mathcal{H}_1(F) \xleftarrow{V} \mathcal{H}_{-1}(F) \xleftarrow{\mathcal{E}_\lambda^\diamond(H_0)} \mathfrak{h}_\lambda(H, F).$$

Here  $\mathcal{E}_\lambda^\diamond(H_0)$  is a modified conjugate of  $\mathcal{E}_\lambda(H_0): \mathcal{H}_1(F) \rightarrow \mathfrak{h}_\lambda(H, F)$  defined by equality

$$\langle \mathcal{E}_\lambda(H_0) f, g \rangle_{\mathfrak{h}_\lambda(H, F)} = \langle f, \mathcal{E}_\lambda^\diamond(H_0) g \rangle_{1, -1} \quad \forall f \in \mathcal{H}_1(F), \quad g \in \mathfrak{h}_\lambda(H, F).$$

**1.3. Description of sections.** In section 2 we give an exposition of sheaves of Hilbert spaces. In section 3 we study self-adjoint operators  $H$  on rigged Hilbert spaces  $(\mathcal{H}, F)$  which are compatible with the rigging  $F$  in the sense that the condition (13) holds; in particular we construct a sheaf  $\mathcal{S}(H, F)$  of Hilbert spaces over an explicitly defined set  $\Lambda(H, F)$  of full Lebesgue measure, associated with a compatible pair  $(H, F)$  and we show that the sheaf  $\mathcal{S}(H, F)$  gives a natural diagonalization of the operator  $H$  (Theorem 3.15). In section 4 we give new definitions of wave matrices (57) and wave operators (65), prove unitarity (Corollary 4.5) and the multiplicative property (Theorem 4.4) of the wave



matrices and show that these definitions coincide with classical time-dependent definitions (Theorem 4.10). In section 5 we give new definitions of the scattering matrix (67) and the scattering operator (68) and give a new proof of the stationary formula for the scattering matrix (Theorem 5.2). We also show that thus introduced notions of scattering theory possess many other well-known properties, e.g. Theorems 4.7 and 5.1. Finally, in Section 6 we give an example of a class of Schrödinger operators, to which the results of Sections 4 and 5 are applied.

## 2. SHEAVES OF HILBERT SPACES

The notion of a sheaf was introduced by A. Grothendieck with the aim to give a general coordinate-independent definition of algebraic variety. Sheaves of rings and vectors spaces are used extensively in topology and geometry, see for instance [Shaf] and [B]. Our approach to scattering theory uses sheaves of Hilbert spaces. Since I was not able to find an appropriate reference on sheaves of Hilbert spaces, which would satisfy needs of this paper, this section is devoted to an exposition of this notion.

Before proceeding to this exposition we note that reasons for using sheaves in geometry and in this paper are different. In topology sheaves are used because of and for the study of non-trivial homotopical and homological structure of underlying topological space. In this paper we use sheaves of Hilbert spaces over a certain subset  $\Lambda$  of  $\mathbb{R}$  which has full Lebesgue measure and topological structure of this set is not of interest. The need in such sheaves arises here since we are able to construct certain direct integrals of fiber Hilbert spaces over all bounded open subsets of  $\Lambda$ , but not over all  $\Lambda$  and therefore we need to “glue” together the direct integrals over intersecting bounded subsets.

**2.1. A sheaf of fiber Hilbert spaces.** Let  $\Lambda$  be a topological space with a fixed base  $\mathcal{B}$  of topology. In addition, later we assume that  $\mathcal{B}$  contains intersection of any two sets from  $\mathcal{B}$  as long as this intersection is not empty and that the space  $\Lambda$  is a union of an increasing family of sets  $\Delta_1 \subset \Delta_2 \subset \dots$  from  $\mathcal{B}$ .

Let  $\lambda \in \Lambda$ . By  $\mathcal{B}_\lambda$  we denote the subset  $\{\Delta \in \mathcal{B} : \lambda \in \Delta\}$  of  $\mathcal{B}$ . A *sheaf of fiber Hilbert spaces* (or fiber of a sheaf of Hilbert spaces)  $\mathfrak{h}_\lambda$  at  $\lambda$  is a collection of Hilbert spaces

$$\mathfrak{h}_\lambda := \{\mathfrak{h}_\lambda(\Delta) : \Delta \in \mathcal{B}_\lambda\}$$

and a collection of unitary isomorphisms

$$\left\{ U_{\Delta_2, \Delta_1}(\lambda) : \mathfrak{h}_\lambda(\Delta_1) \xrightarrow{\sim} \mathfrak{h}_\lambda(\Delta_2) \mid (\Delta_1, \Delta_2) \in \mathcal{B}_\lambda^2 \right\}$$

such that for any three, not necessarily distinct, open subsets  $\Delta_1, \Delta_2$  and  $\Delta_3$  from  $\mathcal{B}_\lambda$  we have the equality

$$(14) \quad U_{\Delta_3, \Delta_1}(\lambda) = U_{\Delta_3, \Delta_2}(\lambda) U_{\Delta_2, \Delta_1}(\lambda).$$

It follows from this that for any  $\Delta_1, \Delta_2, \Delta \in \mathcal{B}_\lambda$  the equalities

$$U_{\Delta_2, \Delta_1}(\lambda)^* = U_{\Delta_1, \Delta_2}(\lambda) \text{ and } U_{\Delta, \Delta}(\lambda) = 1_{\mathfrak{h}_\lambda(\Delta)}$$

hold, where  $1_{\mathfrak{h}_\lambda(\Delta)}$  is the identity operator on  $\mathfrak{h}_\lambda(\Delta)$ . An element of the sheaf  $\mathfrak{h}_\lambda$  is a collection of vectors

$$f(\lambda) = \{f_\Delta(\lambda) \in \mathfrak{h}_\lambda(\Delta) : \Delta \in \mathcal{B}_\lambda\},$$

such that for any  $\Delta_1, \Delta_2 \in \mathcal{B}_\lambda$  there holds the equality

$$(15) \quad U_{\Delta_2, \Delta_1}(\lambda) f_{\Delta_1}(\lambda) = f_{\Delta_2}(\lambda).$$

A sheaf  $\mathfrak{h}_\lambda$  of fiber Hilbert spaces at  $\lambda \in \Lambda$  is a vector space with scalar product

$$(16) \quad \langle f(\lambda), g(\lambda) \rangle_{\mathfrak{h}_\lambda} = \langle f_\Delta(\lambda), g_\Delta(\lambda) \rangle_{\mathfrak{h}_\lambda(\Delta)},$$

where  $\Delta$  is any element of  $\mathcal{B}_\lambda$ . Plainly, this scalar product does not depend on the choice of  $\Delta$ . It is equally obvious that  $\mathfrak{h}_\lambda$  is a Hilbert space. In what follows, the Hilbert spaces  $\mathfrak{h}_\lambda(\Delta)$  will usually be subspaces of a single Hilbert space  $\mathfrak{h}$ , namely  $\mathfrak{h} = \ell_2$ . In this case we say that  $\mathfrak{h}_\lambda$  is a sheaf of fiber Hilbert spaces in  $\mathfrak{h}$ .

**2.1.1. Operators acting on a sheaf of fiber Hilbert spaces.** Let  $\mathcal{K}$  be a Hilbert space. An operator  $T$  from a Hilbert space  $\mathcal{K}$  to  $\mathfrak{h}_\lambda$  is a family of operators  $\{T_\Delta : \mathcal{K} \rightarrow \mathfrak{h}_\lambda(\Delta), \Delta \in \mathcal{B}_\lambda\}$  such that for any  $\Delta_1, \Delta_2 \in \mathcal{B}_\lambda$  the equality

$$(17) \quad T_{\Delta_2} = U_{\Delta_2, \Delta_1}(\lambda) T_{\Delta_1}$$

holds. An operator  $T$  from  $\mathfrak{h}_\lambda$  to a Hilbert space  $\mathcal{K}$  is a family of operators

$$\{T_\Delta : \mathfrak{h}_\lambda(\Delta) \rightarrow \mathcal{K}, \Delta \in \mathcal{B}_\lambda\}$$

such that for any  $\Delta_1, \Delta_2 \in \mathcal{B}_\lambda$  and any  $f(\lambda) \in \mathfrak{h}_\lambda$  there holds the equality

$$T_{\Delta_2}(f_{\Delta_2}(\lambda)) = T_{\Delta_1}(f_{\Delta_1}(\lambda)).$$

This condition is equivalent to this one:

$$T_{\Delta_2} U_{\Delta_2, \Delta_1}(\lambda) = T_{\Delta_1}.$$

**2.2. A sheaf of Hilbert spaces.** Let  $\{\mathcal{H}_\lambda, \lambda \in \Delta\}$  be a family of subspaces of a Hilbert space  $\mathcal{H}$ . We say that this family is *measurable*, if the family of orthogonal projection  $P_\lambda$  onto  $\mathcal{H}_\lambda$  is measurable, that is, if for any  $f, g \in \mathcal{H}$ , the function  $\Delta \ni \lambda \mapsto \langle f, P_\lambda g \rangle$  is measurable. A *measurable section* of this family is a measurable function  $f : \Delta \rightarrow \mathcal{H}$  such that  $f(\lambda) \in \mathcal{H}_\lambda$  for all  $\lambda \in \Delta$ .

Let  $\mathfrak{h}$  be a Hilbert space, let  $\Lambda$  be a Hausdorff topological space with a fixed base of topology  $\mathcal{B}$ , such that if  $\Delta_1, \Delta_2 \in \mathcal{B}$  then  $\Delta_1 \cap \Delta_2 \in \mathcal{B}$ , and let  $\rho$  be a Borel measure in  $\Lambda$ . A *sheaf of Hilbert spaces*  $\mathcal{S}$  over  $\Lambda$  is a family of sheaves of fiber Hilbert spaces  $\{\mathfrak{h}_\lambda : \lambda \in \Lambda\}$  in  $\mathfrak{h}$ , such that (1) for every  $\Delta \in \mathcal{B}$  the family of Hilbert spaces  $\{\mathfrak{h}_\lambda(\Delta) : \lambda \in \Delta\}$  is measurable and (2) for any  $\Delta_1, \Delta_2 \in \mathcal{B}$  the family

$$U_{\Delta_2, \Delta_1} = \{U_{\Delta_2, \Delta_1}(\lambda), \lambda \in \Delta_1 \cap \Delta_2\}$$

is also measurable, that is, it maps measurable sections of  $\{\mathfrak{h}_\lambda(\Delta_1) : \lambda \in \Delta_1 \cap \Delta_2\}$  to measurable sections of  $\{\mathfrak{h}_\lambda(\Delta_2) : \lambda \in \Delta_1 \cap \Delta_2\}$ . It follows that for every  $\Delta \in \mathcal{B}$  the dimension function

$$\Delta \ni \lambda \mapsto \dim(\mathfrak{h}_\lambda(\Delta)) \in \{0, 1, 2, \dots, \infty\}$$

is measurable. Further, to every  $\Delta \in \mathcal{B}$  we can assign a Hilbert space

$$\mathcal{H}(\Delta) := \int_{\Delta}^{\oplus} \mathfrak{h}_{\lambda}(\Delta) \rho(d\lambda),$$

— the direct integral of fiber Hilbert spaces  $\mathfrak{h}_{\lambda}(\Delta)$ . Elements of  $\mathcal{H}(\Delta)$  are measurable square integrable sections of the family  $\{\mathfrak{h}_{\lambda}(\Delta) : \lambda \in \Delta\}$ , where two sections are identified if they coincide for  $\rho$ -a.e.  $\lambda \in \Lambda$ . Definition of the scalar product in  $\mathcal{H}(\Delta)$  is obvious. The operator

$$U_{\Delta_2, \Delta_1} = \int_{\Delta_1 \cap \Delta_2}^{\oplus} U_{\Delta_2, \Delta_1}(\lambda) \rho(d\lambda)$$

is a unitary isomorphism of Hilbert spaces  $\mathcal{H}(\Delta_1)|_{\Delta_1 \cap \Delta_2}$  and  $\mathcal{H}(\Delta_2)|_{\Delta_1 \cap \Delta_2}$ , where

$$\mathcal{H}(\Delta_j)|_{\Delta_1 \cap \Delta_2} = \int_{\Delta_1 \cap \Delta_2}^{\oplus} \mathfrak{h}_{\lambda}(\Delta_j) \rho(d\lambda), \quad j = 1, 2.$$

A sheaf  $\mathcal{S}$  of Hilbert spaces can be given a structure of Hilbert space, the construction of which follows. A *measurable section*  $f$  of the sheaf  $\mathcal{S}$  is a family  $f(\lambda)$  of elements of fiber Hilbert spaces  $\mathfrak{h}_{\lambda}$ , given for  $\rho$ -a.e.  $\lambda$ , such that for any  $\Delta \in \mathcal{B}$  the section  $f_{\Delta}(\cdot)$  of the family  $\{\mathfrak{h}_{\lambda}(\Delta) : \lambda \in \Delta\}$  is measurable. A section  $f$  is *square integrable*, if the number

$$(18) \quad \|f\|_{\mathcal{S}} := \lim_{n \rightarrow \infty} \|f_{\Delta_n}\|_{\mathcal{H}(\Delta_n)} = \sup_{\Delta \in \mathcal{B}} \|f_{\Delta}\|_{\mathcal{H}(\Delta)}$$

is finite, where  $\Delta_1 \subset \Delta_2 \subset \dots$  is an increasing sequence of elements of  $\mathcal{B}$  such that  $\bigcup_{n=1}^{\infty} \Delta_n = \Lambda$ . Obviously, this definition does not depend on the choice of the sequence  $(\Delta_n)_{n=1}^{\infty}$ . The set of all square integrable sections of  $\mathcal{S}$  is a vector space in an obvious way, where as usual we identify two sections which coincide for  $\rho$ -a.e.  $\lambda \in \Lambda$ . We denote this vector space by the same symbol  $\mathcal{S}$ . We also denote an element  $\{f(\lambda) : \lambda \in \Lambda\}$  of  $\mathcal{S}$  by

$$(19) \quad \int_{\Lambda}^{\oplus} f(\lambda) \rho(d\lambda).$$

The scalar product of two square integrable sections  $f, g \in \mathcal{S}$  is defined by the formula

$$\langle f, g \rangle_{\mathcal{S}} := \lim_{n \rightarrow \infty} \langle f_{\Delta_n}, g_{\Delta_n} \rangle_{\mathcal{H}(\Delta_n)},$$

where  $\Delta_1 \subset \Delta_2 \subset \dots$  is a sequence as above. This scalar product is well-defined in the sense that it does not depend on the choice of the sequence  $\Delta_1 \subset \Delta_2 \subset \dots$  of elements of  $\mathcal{B}$ .

**Theorem 2.1.**  *$\mathcal{S}$  is a Hilbert space.*

*Proof.* Let  $f_1, f_2, \dots \in \mathcal{S}$  be a Cauchy sequence. Plainly, for any open set  $\Delta \in \mathcal{B}$  the sequence  $f_{1, \Delta}, f_{2, \Delta}, \dots$  is also Cauchy. Since  $\mathcal{H}(\Delta)$  is complete, the last sequence converges to some  $f_{\Delta} \in \mathcal{H}(\Delta)$ . Since  $U_{\Delta_2, \Delta_1}(\lambda)$  is continuous, we have for a.e.  $\lambda$

$$\begin{aligned} U_{\Delta_2, \Delta_1}(\lambda) f_{\Delta_1}(\lambda) &= U_{\Delta_2, \Delta_1}(\lambda) \lim_{n \rightarrow \infty} f_{n, \Delta_1}(\lambda) \\ &= \lim_{n \rightarrow \infty} U_{\Delta_2, \Delta_1}(\lambda) f_{n, \Delta_1}(\lambda) = \lim_{n \rightarrow \infty} f_{n, \Delta_2}(\lambda) = f_{\Delta_2}(\lambda). \end{aligned}$$

It follows that  $f = \{f_\Delta\}$  defines a measurable section of  $\mathcal{S}$ . We have

$$\begin{aligned} \|f\|_{\mathcal{S}}^2 &= \sup_{k \in \mathbb{N}} \|f_{\Delta_k}\|_{\mathcal{H}(\Delta_k)}^2 = \sup_{k \in \mathbb{N}} \lim_{n \rightarrow \infty} \|f_{n, \Delta_k}\|_{\mathcal{H}(\Delta_k)}^2 \\ &\leq \lim_{n \rightarrow \infty} \sup_{k \in \mathbb{N}} \|f_{n, \Delta_k}\|_{\mathcal{H}(\Delta_k)}^2 = \lim_{n \rightarrow \infty} \|f_n\|_{\mathcal{S}}^2 < \infty, \end{aligned}$$

where the last inequality follows from  $(f_n)$  being Cauchy. It follows that  $f \in \mathcal{S}$ . Now, we show that  $f_n$  converges to  $f$ . Since  $(f_n)$  is Cauchy, it follows from  $|\|f_n(\lambda)\| - \|f(\lambda)\|| \leq \|f_n(\lambda) - f(\lambda)\|$  that the sequence of functions  $\lambda \mapsto (\|f_n(\lambda)\|_{\mathfrak{h}_\lambda})$  is Cauchy in  $L^2(\Lambda, d\rho)$ . By construction,  $\|f_n(\lambda) - f(\lambda)\|_{\mathfrak{h}_\lambda} \rightarrow 0$  for  $\rho$ -a.e.  $\lambda \in \Lambda$ . It follows that  $\|f_n(\lambda)\|_{\mathfrak{h}_\lambda} \rightarrow \|f(\lambda)\|_{\mathfrak{h}_\lambda}$  as functions of  $\lambda$  in  $L^2(\Lambda, d\rho)$ . This implies that for any  $\varepsilon > 0$  there exists an open set  $\Delta \subset \Lambda$  and a number  $N_1$  such that for all  $n \geq N_1$  we have

$$\int_{\Lambda \setminus \Delta} \|f_n(\lambda)\|^2 d\rho(\lambda) < \varepsilon/4 \quad \text{and} \quad \int_{\Lambda \setminus \Delta} \|f(\lambda)\|^2 d\rho(\lambda) < \varepsilon/4.$$

Further, for some  $N_2$  and all  $n \geq N_2$  we have  $\|f_{n, \Delta} - f_\Delta\| < \varepsilon/2$ . It follows that  $f_n$  converges to  $f$  in  $\mathcal{S}$ .  $\square$

The *support* of an element  $f$  of the Hilbert space  $\mathcal{S}$  is defined by formula  $\text{supp } f = \bigcup_{\Delta} \text{supp } f_\Delta$ , where the union is over all open sets  $\Delta \in \mathcal{B}$ . Elements of the sheaf Hilbert space  $\mathcal{S}$  can be represented by either a measurable square integrable family (19) or by a family of vectors  $f_\Delta \in \mathcal{H}(\Delta)$  which satisfy the gluing property (15) for all  $\Delta_1$  and  $\Delta_2$  and for a.e.  $\lambda \in \Delta_1 \cap \Delta_2$  and such that the supremum in (18) is finite.

**2.2.1. Operators acting on a sheaf of Hilbert spaces.** Let  $\mathcal{K}$  be a Hilbert space. As long as definition of sheaf Hilbert space  $\mathcal{S}$  is given, a standard definition of an operator acting from  $\mathcal{K}$  to  $\mathcal{S}$  applies. But in practice there are several equivalent ways to define such an operator. In order to define an operator  $T$  from  $\mathcal{K}$  to the sheaf Hilbert space  $\mathcal{S}$  with domain  $\mathcal{D} \subset \mathcal{K}$  one can present for  $\rho$ -a.e.  $\lambda \in \Lambda$  and for all  $\Delta \in \mathcal{B}_\lambda$  an operator

$$T_\Delta(\lambda): \mathcal{D} \rightarrow \mathfrak{h}_\lambda(\Delta),$$

such that for  $\rho$ -a.e.  $\lambda \in \Lambda$ , for any  $\Delta_1, \Delta_2 \in \mathcal{B}_\lambda$  and for any  $f \in \mathcal{D}$  there holds the equality

$$U_{\Delta_2, \Delta_1}(\lambda) T_{\Delta_1}(\lambda) f = T_{\Delta_2}(\lambda) f,$$

and such that the mapping  $\Delta \ni \lambda \mapsto T_\Delta(\lambda) f \in \mathfrak{h}_\lambda(\Delta)$  is measurable. Another way to define an operator  $\mathcal{K} \rightarrow \mathcal{S}$  is to assign to every  $f \in \mathcal{K}$  an element  $Tf = \{(Tf)_\Delta \in \mathcal{H}(\Delta): \Delta \in \mathcal{B}\}$  of  $\mathcal{S}$ . A family of operators  $T(\lambda): \mathcal{K} \rightarrow \mathfrak{h}_\lambda$  is measurable, if for any  $f \in \mathcal{K}$  the section  $T(\lambda)f$  of the family  $\{\mathfrak{h}_\lambda\}$  is measurable. Given a measurable family of operators  $T(\lambda): \mathcal{K} \rightarrow \mathfrak{h}_\lambda$ , one can define an operator

$$T = \int_{\Lambda}^{\oplus} T(\lambda) \rho(d\lambda): \mathcal{K} \rightarrow \mathcal{S}.$$

## 3. SELF-ADJOINT OPERATORS ON RIGGED HILBERT SPACES

Given a self-adjoint operator  $H_0$  on a Hilbert space  $\mathcal{H}$  and a self-adjoint perturbation  $V$ , our aim is to construct explicitly the wave matrix  $w_{\pm}(\lambda; H_0 + V, H_0)$  for all real numbers  $\lambda$  from some explicitly given set of full Lebesgue measure  $\Lambda \subset \mathbb{R}$ . In order to do this, we need to impose some additional structure. In case of trace-class perturbations  $V$  this additional structure is a Hilbert-Schmidt rigging operator  $F$  (see [Az]). Given a Hilbert-Schmidt rigging  $F$ , for any self-adjoint operator  $H_0$  one can define the set of full Lebesgue measure  $\Lambda(H_0, F)$ , such that for all  $\lambda \in \Lambda(H_0, F) \cap \Lambda(H_0 + V, F)$  it is possible to define the fiber Hilbert space  $\mathfrak{h}_{\lambda}$ , the wave matrices  $w_{\pm}(\lambda; H_0 + V, H_0)$  etc (see [Az]).

In case of non-compact perturbations  $V$ , we need to generalize the notion of the Hilbert-Schmidt rigging operator  $F$ . In the trace-class case, the perturbation  $V$  admits the factorization  $V = F^* J F$ , where  $J$  is any bounded operator. We keep this factorization in the generalization of the trace-class theory which covers the case of non-compact perturbations  $V$ . We also assume that in the factorization  $V = F^* J F$  the operator  $J$  is still allowed to be any bounded operator on the auxiliary Hilbert space  $\mathcal{K}$ . This implies that the rigging operator  $F$  can no longer be assumed to be compact. Further, since the wave matrix cannot exist for general pairs  $(H_0, H_0 + V)$ , this also means that restrictions, which ensure existence of the wave matrices, shift from the perturbation  $V$  to the rigging operator  $F$ . Thus, we need to impose some conditions on the pairs  $(H_j, F)$ ,  $j = 0, 1$ . It turns out that we need only one condition: the operators  $F E_{\Delta}^{H_j}$ ,  $j = 0, 1$ , are Hilbert-Schmidt for any bounded open set  $\Delta$ . In this section we study the pairs  $(H, F)$  of operators which satisfy this condition.

**3.1. Generalized rigging.** Let  $\mathcal{H}$  be a Hilbert space (all Hilbert spaces in this paper are complex and separable, but not necessarily infinite dimensional). The Hilbert space  $\mathcal{H}$  is the main Hilbert space on which operators act. In addition to  $\mathcal{H}$ , we use an auxiliary Hilbert space  $\mathcal{K}$ . By  $\text{dom}(T)$  we denote domain of an operator  $T$ .

**Definition 3.1.** *A (generalized) rigging operator  $F$  on  $\mathcal{H}$  is a bounded operator  $F: \mathcal{H} \rightarrow \mathcal{K}$  with trivial kernel and dense range.*

**Remark 3.2.** That the range of  $F$  is dense in  $\mathcal{K}$  is not essential: one can always replace  $\mathcal{K}$  by the closure of the image of  $F$ . But this is convenient; otherwise, we would need for instance to write  $g \in \overline{\text{ran } F}$  instead of  $g \in \mathcal{K}$ . Sometimes riggings with not dense ranges appear naturally; in such cases we assume that the auxiliary Hilbert space  $\mathcal{K}$  changes appropriately.

Our aim is to study operators  $H$  which act on  $\mathcal{H}$  with a predefined rigging  $F$ . Firstly, we shall consider objects which already can be associated with the pair  $(\mathcal{H}, F)$  (all of them are well-known). A rigging operator  $F$  admits polar decomposition  $F = U |F|$ , where the self-adjoint operator  $|F|$  has trivial kernel and (therefore) dense range, so that the inverse operator  $|F|^{-1}$  exists as an unbounded operator. Therefore, we have a scale of Hilbert

spaces  $\mathcal{H}_\alpha(F)$ ,  $\alpha \in \mathbb{R}$ , introduced as follows: let

$$\mathcal{H}_\infty(F) = \left\{ f \in \mathcal{H} : f \in \text{dom}(|F|^k) \text{ for all } k \in \mathbb{Z} \right\},$$

then  $\mathcal{H}_\alpha$  is the completion of  $\mathcal{H}_\infty$  endowed with scalar product

$$\langle f, g \rangle_\alpha = \langle |F|^{-\alpha} f, |F|^{-\alpha} g \rangle.$$

Since we have assumed  $F$  to be a bounded operator, for  $\alpha > \beta$  there is a natural inclusion  $\mathcal{H}_\alpha(F) \hookrightarrow \mathcal{H}_\beta(F)$ . Hilbert spaces  $\mathcal{H}_\alpha$  and  $\mathcal{H}_\beta$  are isomorphic, and the operator  $|F|^{\alpha-\beta} : \mathcal{H}_\beta(F) \rightarrow \mathcal{H}_\alpha(F)$  is an isomorphism. Hence, the expression  $|F|$  can be understood in two different ways: as an operator on  $\mathcal{H}$ , or as an isomorphism of Hilbert spaces  $\mathcal{H}_\alpha$  and  $\mathcal{H}_{\alpha+1}$ . Since the operator  $U$  from the polar decomposition of  $F$  is unitary (assuming that  $\mathcal{K} = \overline{\text{ran } F}$ ), the operator  $F$  can also be treated as a natural isomorphism of  $\mathcal{H}_{-1}$  and  $\mathcal{K}$ , and the operator  $F^*$  can be treated as a natural isomorphism of  $\mathcal{K}$  and  $\mathcal{H}_1$ . The same remark relates to any power of  $|F|$ .

For any real number  $\alpha$  there is a natural pairing  $\langle \cdot, \cdot \rangle_{\alpha, -\alpha} : \mathcal{H}_\alpha \times \mathcal{H}_{-\alpha} \rightarrow \mathbb{C}$ , such that for any  $f, g \in \mathcal{H}_\alpha \cap \mathcal{H}_{-\alpha}$  we have

$$\langle f, g \rangle_{\alpha, -\alpha} = \langle f, g \rangle.$$

In the sequel we shall need only Hilbert spaces  $\mathcal{H}_1(F)$  and  $\mathcal{H}_{-1}(F)$ . For this reason, further definitions are given only for this case of  $\alpha = \pm 1$ . Elements of the Hilbert space  $\mathcal{H}_1(F)$  are to be considered as smooth or regular vectors of the Hilbert space  $\mathcal{H}$ , while elements of  $\mathcal{H}_{-1}(F)$  are considered as singular (improper) vectors, which may not lie in the main Hilbert space  $\mathcal{H}$ . If  $\tilde{\mathcal{H}}$  is another Hilbert space and  $A : \mathcal{H}_1 \rightarrow \tilde{\mathcal{H}}$  is a bounded operator, then there is a unique bounded operator  $A^\diamond : \tilde{\mathcal{H}} \rightarrow \mathcal{H}_{-1}$ , such that for any  $f \in \mathcal{H}_1$  and  $g \in \tilde{\mathcal{H}}$  there holds the equality

$$(20) \quad \langle Af, g \rangle_{\tilde{\mathcal{H}}} = \langle f, A^\diamond g \rangle_{-1, 1}.$$

The operator  $A^\diamond$  can also be defined by formula  $A^\diamond = |F|^{-2} A^*$ , where  $|F|^{-2}$  is understood as an isomorphism of  $\mathcal{H}_1(F)$  and  $\mathcal{H}_{-1}(F)$ . We shall have an opportunity to use the following formula which holds for any  $\alpha \in \mathbb{R}$ :

$$(21) \quad \|f\|_{\mathcal{H}_{-\alpha}(F)} = \sup_{\|g\|_{\mathcal{H}_\alpha(F)}=1} \left| \langle f, g \rangle_{-\alpha, \alpha} \right|.$$

**3.2. Compatible pairs  $(H, F)$ .** Throughout this paper we shall assume that  $F$  is a bounded operator from  $\mathcal{H}$  to  $\mathcal{K}$  with trivial kernel. We shall assume that  $F$  also has dense range though it is not necessary. We shall call  $F$  a rigging operator.

**Definition 3.3.** *Let  $H$  be a self-adjoint operator on the Hilbert space  $\mathcal{H}$  with rigging  $F$ . The operator  $H$  is compatible with rigging  $F$ , if for all bounded open subsets  $\Delta$  of  $\mathbb{R}$  the operator*

$$(22) \quad F_\Delta := FE_\Delta \text{ is Hilbert-Schmidt.}$$

Proof of the following lemma is standard and well-known, but we give it for completeness.

**Lemma 3.4.** *If  $H$  is a self-adjoint operator compatible with a rigging  $F$ , then for all non-real  $z$  the operator  $FR_z(H)F^*$  is compact.*

*Proof.* The operator  $FR_z(H)F^*$  can be approximated in uniform norm arbitrarily well by a compact operator  $FE_{[-a,a]}^H R_z(H)F^*$  where  $a$  is large enough number.  $\square$

Given a compatible pair  $(H, F)$  of operators one can introduce a sheaf of Hilbert spaces  $\mathcal{S}(H, F)$  over some measurable set  $\Lambda(H, F) \subset \mathbb{R}$  of full Lebesgue measure, naturally associated with the pair  $(H, F)$ ; the main property of this sheaf is that it diagonalizes the operator  $H$ . The rest of this section is devoted to this construction.

**3.3. Hilbert spaces  $\mathcal{H}_{\pm 1}(\Delta)$ .** Given a self-adjoint operator  $H$  on a rigged Hilbert space  $(\mathcal{H}, F)$ , such that  $H$  is compatible with the rigging  $F$ , we can further study some properties of Hilbert spaces  $\mathcal{H}_1(F)$  and  $\mathcal{H}_{-1}(F)$ , introduced in Section 3.1. Given a bounded open set  $\Delta$ , we can consider the Hilbert-Schmidt operator  $F_\Delta = FE_\Delta$  as a Hilbert-Schmidt rigging in the Hilbert space  $E_\Delta \mathcal{H}$ . Corresponding Hilbert spaces  $\mathcal{H}_{\pm 1}$  of regular and singular vectors will be denoted by  $\mathcal{H}_{\pm 1}(\Delta)$ . Let

$$(23) \quad F_\Delta = \sum_{j=1}^{\infty} \kappa_j^\Delta \langle \varphi_j^\Delta, \cdot \rangle \psi_j^\Delta$$

be a fixed Schmidt representation of  $F_\Delta$ . Elements of  $\mathcal{H}_1(\Delta)$  have the form

$$(24) \quad f = \sum_{j=1}^{\infty} \beta_j \kappa_j^\Delta \varphi_j^\Delta$$

where  $\beta = (\beta_j) \in \ell_2$  and

$$(25) \quad \|f\|_{\mathcal{H}_1(\Delta)} = \|\beta\|_{\ell_2}.$$

The one-to-one correspondence between elements  $f$  of  $\mathcal{H}_1(\Delta)$  and  $\ell_2$ -vectors  $\beta = (\beta_j)$  given by (24) is a unitary operator. For this reason, for an element  $f$  of  $\mathcal{H}_1(\Delta)$  we use notation  $\beta_j^\Delta(f)$  so that, by definition,

$$(26) \quad f = \sum_{j=1}^{\infty} \beta_j^\Delta(f) \kappa_j^\Delta \varphi_j^\Delta, \quad \beta^\Delta(f) \in \ell_2.$$

We have from (23)

$$(27) \quad F_\Delta^* = \sum_{j=1}^{\infty} \kappa_j^\Delta \langle \psi_j^\Delta, \cdot \rangle \varphi_j^\Delta.$$

It follows that the range of  $F_\Delta^*$  consists of all vectors of the form (24), that is,

$$(28) \quad \mathcal{H}_1(\Delta) = \text{ran}(F_\Delta^*).$$

The inclusion  $E_\Delta f \in \mathcal{H}_1(\Delta)$  means that restriction  $E_\Delta f$  of a vector  $f \in \mathcal{H}$  to a bounded Borel set  $\Delta$  is regular in a certain sense. It is natural to expect that if a vector  $f$  is regular on some Borel set  $\Delta$ , then  $f$  is to be regular also on any Borel subset  $\Delta_0$  of the set  $\Delta$ . One could compare this situation to continuity property of a mapping: if a mapping is continuous on a set, then it is also continuous on a subset of the set. The following

proposition asserts that regularity of a vector  $f$  on  $\Delta$  is inherited by subsets of  $\Delta$ , but this is where the analogy with continuity of functions ends, since the restriction mapping turns out to be surjective.

**Proposition 3.5.** *Let  $\Delta_1 \subset \Delta_2$  be two open bounded subsets of  $\mathbb{R}$ . If  $f \in \mathcal{H}_1(\Delta_2)$ , then  $E_{\Delta_1}f \in \mathcal{H}_1(\Delta_1)$ . Moreover, the mapping  $E_{\Delta_1} : \mathcal{H}_1(\Delta_2) \rightarrow \mathcal{H}_1(\Delta_1)$  is a contraction and its image coincides with  $\mathcal{H}_1(\Delta_1)$ .*

*Proof.* If  $f \in \mathcal{H}_1(\Delta_2)$ , then, by (28), there exists  $g \in \mathcal{K}$  such that  $f = E_{\Delta_2}F^*g$ . It follows that

$$E_{\Delta_1}f = E_{\Delta_1}F^*g = F_{\Delta_1}^*g \in \mathcal{H}_1(\Delta_1),$$

where the last inclusion follows from (28). Since, again by (28), the set  $\mathcal{H}_1(\Delta)$  is the image of the operator  $F_{\Delta}^* = E_{\Delta}F^*$ , it follows that  $E_{\Delta_1}\mathcal{H}_1(\Delta_2)$  coincides with  $\mathcal{H}_1(\Delta_1)$ .

Direct calculation shows that for any  $\Delta_1 \subset \Delta_2$  the following formula holds

$$(29) \quad \beta_j^{\Delta_1}(E_{\Delta_1}f) = \sum_{k=1}^{d(\Delta_2)} \langle \psi_j^{\Delta_1}, \psi_k^{\Delta_2} \rangle \beta_k^{\Delta_2}(f),$$

where  $d(\Delta_2)$  is the dimension of the Hilbert space  $E_{\Delta_2}\mathcal{H}$ . Letting

$$\Psi^{\Delta_1, \Delta_2} = (\langle \psi_j^{\Delta_1}, \psi_k^{\Delta_2} \rangle)_{j=1, k=1}^{d(\Delta_1), d(\Delta_2)},$$

we can rewrite (29) as

$$(30) \quad \beta^{\Delta_1}(E_{\Delta_1}f) = \Psi^{\Delta_1, \Delta_2} \beta^{\Delta_2}(f).$$

The matrix  $\Psi^{\Delta_1, \Delta_2}$  is a contraction. Indeed,  $\Psi^{\Delta_1, \Delta_2}$  is the matrix of the orthogonal projection of the linear span of  $(\psi_j^{\Delta_2})$  onto the linear span of  $(\psi_j^{\Delta_1})$ , with the orthonormal bases  $(\psi_j^{\Delta_2})_{j=1}^{d(\Delta_2)}$  and  $(\psi_j^{\Delta_1})_{j=1}^{d(\Delta_1)}$  respectively. That the operator  $E_{\Delta_1} : \mathcal{H}_1(\Delta_2) \rightarrow \mathcal{H}_1(\Delta_1)$  is a contraction now follows from (30), (25) and from the fact that the matrix  $\Psi^{\Delta_1, \Delta_2}$  is a contraction.  $\square$

Clearly, for any  $\Delta_1 \subset \Delta_2 \subset \Delta_3$ , we have the equality

$$\Psi^{\Delta_1, \Delta_3} = \Psi^{\Delta_1, \Delta_2} \Psi^{\Delta_2, \Delta_3}.$$

The formula (30) can be rewritten as

$$(31) \quad E_{\Delta_1} \left( \sum_{k=1}^{d(\Delta_2)} \beta_k^{\Delta_2} \kappa_k^{\Delta_2} \varphi_k^{\Delta_2} \right) = \left( \sum_{j=1}^{d(\Delta_1)} (\Psi^{\Delta_1, \Delta_2} \beta^{\Delta_2})_j \kappa_j^{\Delta_1} \varphi_j^{\Delta_1} \right).$$

**Lemma 3.6.** *For any  $f \in \mathcal{H}_1(F)$  we have  $E_{\Delta}f \in \mathcal{H}_1(\Delta)$  and*

$$\|f\|_{\mathcal{H}_1(F)} = \lim_{\Delta \rightarrow \mathbb{R}} \|E_{\Delta}f\|_{\mathcal{H}_1(\Delta)}.$$

*Proof.* Since  $\mathcal{H}_1(F)$  and  $\text{ran } F^*$  coincide as sets, by (28) the inclusion  $f \in \mathcal{H}_1(F)$  implies that  $E_{\Delta}f \in \mathcal{H}_1(\Delta)$ . In particular, the equality to be proved makes sense. Further, there exists a unique vector  $g \in \mathcal{K}$  such that  $f = F^*g$ , and, by definition,

$$(32) \quad \|f\|_{\mathcal{H}_1(F)} = \|g\|_{\mathcal{K}}.$$



Hence,  $E_\Delta f = F_\Delta^* g$ . Since the operator  $F_\Delta^*$  has the form (27), it follows that

$$E_\Delta f = F_\Delta^* g = \sum_{j=1}^{\infty} \kappa_j^\Delta \langle \psi_j^\Delta, g \rangle \varphi_j^\Delta.$$

This and (25) imply that

$$\|E_\Delta f\|_{\mathcal{H}_1(\Delta)} = \|(\langle \psi_j^\Delta, g \rangle)\|_{\ell_2}.$$

If we denote by  $E_\Delta^\psi$  the projection onto the closed linear span of  $(\psi_j^\Delta)$ , we can rewrite the previous equality as

$$(33) \quad \|E_\Delta f\|_{\mathcal{H}_1(\Delta)} = \|E_\Delta^\psi g\|.$$

Since  $E_\Delta^\psi \nearrow 1_K$  as  $\Delta \rightarrow \mathbb{R}$  (indeed,  $E_\Delta^\psi$  projects onto the image of  $E_\Delta \mathcal{H}$  under the mapping  $F$ ), it follows that  $E_\Delta^\psi \rightarrow 1_K$  in the strong operator topology, so that  $\|E_\Delta^\psi g\| \rightarrow \|g\|$  as  $\Delta \rightarrow \mathbb{R}$ . Combining this with (32) and (33) completes the proof.  $\square$

**Lemma 3.7.** *The operator  $E_\Delta$ , considered as an operator from  $\mathcal{H}_1(F)$  to  $\mathcal{H}_1(\Delta)$ , is a contraction. The image of this contraction coincides with  $\mathcal{H}_1(\Delta)$ .*

*Proof.* Let  $f$  be a unit vector from  $\mathcal{H}_1(F)$  and let  $\varepsilon > 0$ . By Lemma 3.6 there exists large enough  $\Delta_2 \supset \Delta$  such that  $\|E_{\Delta_2} f\|_{\mathcal{H}_1(\Delta_2)} < 1 + \varepsilon$ . Since  $E_\Delta f = E_\Delta E_{\Delta_2} f$ , and since, by Proposition 3.5,  $E_\Delta$  is a contraction from  $\mathcal{H}_1(\Delta_2)$  to  $\mathcal{H}_1(\Delta)$ , it follows that

$$\|E_\Delta f\|_{\mathcal{H}_1(\Delta)} < 1 + \varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, it follows that  $E_\Delta$  is a contraction. Further, by (28) the set  $\mathcal{H}_1(\Delta)$  is the image of the operator  $F_\Delta^* = E_\Delta F^*$ , while  $\mathcal{H}_1(F)$  is the image of  $F^*$ . It follows that  $E_\Delta \mathcal{H}_1(F) = \mathcal{H}_1(\Delta)$ .  $\square$

**Proposition 3.8.** *Let  $\Delta_1 \subset \Delta_2$  be two open bounded subsets of  $\mathbb{R}$ . A natural inclusion*

$$\mathcal{H}_{-1}(\Delta_1) \hookrightarrow \mathcal{H}_{-1}(\Delta_2), \quad f_{\Delta_1} \mapsto f_{\Delta_2},$$

*given by formula*

$$\langle f_{\Delta_2}, k_{\Delta_2} \rangle_{\mathcal{H}_{-1}(\Delta_2), \mathcal{H}_1(\Delta_2)} = \langle f_{\Delta_1}, E_{\Delta_1} k_{\Delta_2} \rangle_{\mathcal{H}_{-1}(\Delta_1), \mathcal{H}_1(\Delta_1)},$$

*where  $k_{\Delta_2} \in \mathcal{H}_1(\Delta_2)$ , is a contraction. Further, for any bounded open  $\Delta \subset \mathbb{R}$  the natural inclusion  $\mathcal{H}_{-1}(\Delta) \hookrightarrow \mathcal{H}_{-1}(F)$ ,  $f_\Delta \mapsto f$ , given by formula*

$$\langle f, k \rangle_{-1,1} = \langle f_\Delta, E_\Delta k \rangle_{-1,1},$$

*where  $k \in \mathcal{H}_1(F)$ , is also a contraction. Moreover, the union  $\bigcup_\Delta \mathcal{H}_{-1}(\Delta)$ , taken over all bounded open sets  $\Delta \subset \mathbb{R}$ , is dense in  $\mathcal{H}_{-1}(F)$ .*

*Proof.* We have, by (21),

$$\begin{aligned} \|f_{\Delta_2}\|_{\mathcal{H}_{-1}(\Delta_2)} &= \sup_{g_{\Delta_2}: \|g_{\Delta_2}\|_{\mathcal{H}_{-1}(\Delta_2)} \leq 1} \left| \langle f_{\Delta_2}, g_{\Delta_2} \rangle_{\mathcal{H}_{-1}(\Delta_2)} \right| \\ &= \sup_{g_{\Delta_2}: \|g_{\Delta_2}\|_{\mathcal{H}_{-1}(\Delta_2)} \leq 1} \left| \langle f_{\Delta_2}, |F_{\Delta_2}|^2 g_{\Delta_2} \rangle_{\mathcal{H}_{-1}(\Delta_2), \mathcal{H}_1(\Delta_2)} \right|. \end{aligned}$$

Let  $k_{\Delta_2} = |F_{\Delta_2}|^2 g_{\Delta_2} \in \mathcal{H}_1(\Delta_2)$ . Since, by definition,  $\|g_{\Delta_2}\|_{\mathcal{H}_{-1}(\Delta_2)} = \|k_{\Delta_2}\|_{\mathcal{H}_1(\Delta_2)}$ , it follows that

$$\begin{aligned} \|f_{\Delta_2}\|_{\mathcal{H}_{-1}(\Delta_2)} &= \sup_{k_{\Delta_2}: \|k_{\Delta_2}\|_{\mathcal{H}_1(\Delta_2)} \leq 1} \left| \langle f_{\Delta_2}, k_{\Delta_2} \rangle_{\mathcal{H}_{-1}(\Delta_2), \mathcal{H}_1(\Delta_2)} \right| \\ &= \sup_{k_{\Delta_2}: \|k_{\Delta_2}\|_{\mathcal{H}_1(\Delta_2)} \leq 1} \left| \langle f_{\Delta_1}, E_{\Delta_1} k_{\Delta_2} \rangle_{\mathcal{H}_{-1}(\Delta_1), \mathcal{H}_1(\Delta_1)} \right|, \end{aligned}$$

where the last equality follows from the definition of  $f_{\Delta_2}$ . Since, by Proposition 3.5, the operator  $E_{\Delta_1}: \mathcal{H}_1(\Delta_2) \rightarrow \mathcal{H}_1(\Delta_1)$  is a contraction, it follows from the last inequality and (21) that  $\|f_{\Delta_2}\|_{\mathcal{H}_{-1}(\Delta_2)} \leq \|f_{\Delta_1}\|_{\mathcal{H}_{-1}(\Delta_1)}$ .

Proof of the second part is similar, but instead of Proposition 3.5 one has to use Lemma 3.7. The last assertion is obvious.  $\square$

**3.4. The set  $\Lambda(H, F)$ .** Given a fixed Hilbert-Schmidt rigging operator  $F$ , to every self-adjoint operator  $H$  we can assign a set of full Lebesgue measure  $\Lambda(H, F)$ , so that for every number  $\lambda$  from  $\Lambda(H, F)$  one can define the fiber Hilbert space (6) and the evaluation operator (8) by formula (9). The evaluation operator  $\mathcal{E}_\lambda$  is an important tool in the approach to abstract scattering theory, discussed here. In this section we define and study the set  $\Lambda(H, F)$  for a rigging operator  $F$ , which is not necessarily compact.

**Definition 3.9.** *Let a self-adjoint operator  $H$  on a Hilbert space  $\mathcal{H}$  be compatible with a rigging operator  $F: \mathcal{H} \rightarrow \mathcal{K}$ . The set  $\Lambda(H, F)$  of  $H$ -regular points (or just regular, if there is no danger of confusion) for the pair  $(H, F)$  consists of all those non-zero real numbers  $\lambda$ , such that there exists the norm limit*

$$FR_{\lambda+i0}(H)F^* = \lim_{y \rightarrow 0^+} FR_{\lambda+iy}(H)F^*$$

*and such that for some (and thus for any) bounded open set  $\Delta$ , containing the point  $\lambda$ , the limit*

$$FE_\Delta \operatorname{Im} R_{\lambda+i0}(H)F^* = \lim_{y \rightarrow 0^+} FE_\Delta \operatorname{Im} R_{\lambda+iy}(H)F^*$$

*exists in trace-class norm.*

If  $F$  is a Hilbert-Schmidt operator, then this definition coincides with that of part I of § 1.1 (cf. also [Az, Definition 2.4.1]). Definition of the set  $\Lambda(H, F)$  does not depend on the choice of an open bounded set  $\Delta$  containing  $\lambda$ . Indeed, if  $\Delta_2 \ni \lambda$  is another such set, then  $\lambda$  belongs to the resolvent set of  $E_{\Delta \setminus \Delta_2} R_{\lambda+iy}(H)$ , so that the norm limit  $E_{\Delta \setminus \Delta_2} R_{\lambda+i0}(H)$  exists even without sandwiching by  $F$  and  $F^*$ . Using notation (22), the second condition of Definition 3.9 is equivalent to the existence of the trace class norm limit of the operator

$$F_\Delta \operatorname{Im} R_{\lambda+iy}(H)F_\Delta^*$$

as  $y \rightarrow 0^+$ . It is also not difficult to see that the first condition of Definition 3.9 implies that for any bounded open set  $\Delta$ , containing the point  $\lambda$ , the norm limit

$$F_\Delta R_{\lambda+i0}(H)F_\Delta^* = \lim_{y \rightarrow 0^+} F_\Delta R_{\lambda+iy}(H)F_\Delta^*$$

exists. That is, if  $\lambda$  belongs to the set  $\Lambda(H, F)$  in the sense of Definition 3.9, then for any bounded open set  $\Delta$  containing  $\lambda$ , the point  $\lambda$  belongs to the set  $\Lambda(HE_\Delta, F_\Delta)$  in the sense of part I of § 1.1, where  $HE_\Delta$  and  $F_\Delta$  are considered as operators on the Hilbert space  $E_\Delta \mathcal{H}$ . In fact, if  $\lambda \in \Lambda(HE_\Delta, F_\Delta)$  for some  $\Delta \in \mathcal{B}_\lambda$ , then  $\lambda \in \Lambda(H, F)$  in the sense of Definition 3.9, so that

$$(34) \quad \Lambda(H, F) = \bigcup_{\Delta \in \mathcal{B}} (\Lambda(HE_\Delta, F_\Delta) \cap \Delta).$$

Thus, Definition 3.9 is a natural extension of the definition of the set of regular points  $\Lambda(H, F)$  to the case of generalized rigging operators  $F$ .

**Proposition 3.10.** *The set  $\Lambda(H, F)$  has full Lebesgue measure.*

*Proof.* By Theorem 1.1, the set  $\Lambda(HE_\Delta, F_\Delta)$  has full Lebesgue measure in  $\Delta$ . Further, as it was discussed above, the inclusion

$$(35) \quad \Lambda(HE_{\Delta_1}, F_{\Delta_1}) \cap \Delta_1 \subset \Lambda(HE_{\Delta_2}, F_{\Delta_2}) \cap \Delta_2,$$

holds for any two bounded open subsets  $\Delta_1 \subset \Delta_2$  of  $\mathbb{R}$ . □

We call a null Borel set  $Z$  a *core* of the singular spectrum of  $H$ , if the operator  $E_{\mathbb{R} \setminus Z}^H H$  is absolutely continuous.

**Proposition 3.11.** *The complement of  $\Lambda(H, F)$  is a core of singular spectrum of  $H_0$ .*

*Proof.* It follows from (34) and (35) that

$$\mathbb{R} \setminus \Lambda(H, F) = \bigcap_{\Delta \in \mathcal{B}} (\mathbb{R} \setminus \Lambda(HE_\Delta, F_\Delta)).$$

By Theorem 1.2, for any bounded open set  $\Delta$  the set  $\mathbb{R} \setminus \Lambda(HE_\Delta, F_\Delta)$  is a core of singular spectrum of  $HE_\Delta$ . It follows that  $\mathbb{R} \setminus \Lambda(H, F)$  is a core of the singular spectrum of  $H$ . □

**3.5. A sheaf of fiber Hilbert spaces  $\mathfrak{h}_\lambda(H, F)$ .** Let  $\mathcal{B}$  be the family of all bounded open subsets of  $\Lambda(H, F)$  and let  $\mathcal{B}_\lambda$  be the family of all those bounded open sets which contain a point  $\lambda \in \Lambda(H, F)$ . Our next task is to construct the fiber Hilbert space  $\mathfrak{h}_\lambda$ . In the case of non-compact rigging  $F$ , there is no canonical choice of the fiber Hilbert space  $\mathfrak{h}_\lambda$ . Using the construction of the fiber Hilbert space from part II of § 1.1 for the case of Hilbert-Schmidt rigging  $F$ , once some bounded open set  $\Delta \in \mathcal{B}_\lambda$  is fixed, one can define a fiber Hilbert space  $\mathfrak{h}_\lambda(\Delta)$ , using the fact that the operator  $F_\Delta = FE_\Delta$  is Hilbert-Schmidt. Thus, instead of one fiber Hilbert space  $\mathfrak{h}_\lambda$ , we get a family of Hilbert spaces  $\mathfrak{h}_\lambda(\Delta)$ . Fortunately, it turns out that all these fiber Hilbert spaces are naturally isomorphic; in fact, they form a sheaf of fiber Hilbert spaces at  $\lambda$ , in the sense of subsection 2.1. In order to make the construction of the fiber Hilbert space  $\mathfrak{h}_\lambda$  natural, we treat the Hilbert space  $\mathfrak{h}_\lambda$  as a sheaf of fiber Hilbert spaces at a point. This subsection is devoted to the construction of this sheaf of fiber Hilbert spaces.

Let  $(H, F)$  be a compatible pair (see Definition 3.3). For any bounded open set  $\Delta$ , the operator  $F_\Delta$  is Hilbert-Schmidt (see (22) and (23)) and has zero kernel in  $E_\Delta \mathcal{H}$ .

Hence, the operator  $F_\Delta$  can be considered as a Hilbert-Schmidt rigging in  $E_\Delta \mathcal{H}$ . For any real number  $\lambda$  from the set  $\Lambda(HE_\Delta^H, F_\Delta) \cap \Delta$ , the rigging  $F_\Delta$  generates the evaluation operator (see part IV of § 1.1)

$$\mathcal{E}_\lambda^\Delta: \mathcal{H}_1(\Delta) \rightarrow \ell_2,$$

This operator acts by formula

$$(36) \quad \mathcal{E}_\lambda^\Delta f = \sum_{j=1}^{\infty} \beta_j^\Delta \eta_j^\Delta(\lambda),$$

where  $f$  is an element of the Hilbert space  $\mathcal{H}_1(\Delta)$ , which is given by formula (26), and  $\eta_j^\Delta(\lambda)$  is the  $j$ -th column of the Hilbert-Schmidt matrix

$$(37) \quad \eta^\Delta(\lambda) = \sqrt{\varphi^\Delta(\lambda)},$$

where

$$(38) \quad \begin{aligned} \varphi^\Delta(\lambda) &= \pi^{-1} \left( \kappa_j^\Delta \kappa_k^\Delta \langle \varphi_j^\Delta, \operatorname{Im} R_{\lambda+i0}(E_\Delta^H H) \varphi_k^\Delta \rangle_{1,-1} \right)_{j,k=1}^{\infty} \\ &= \pi^{-1} \left( \langle \psi_j^\Delta, F_\Delta \operatorname{Im} R_{\lambda+i0}(E_\Delta^H H) F_\Delta^* \psi_k^\Delta \rangle_{1,-1} \right)_{j,k=1}^{\infty}. \end{aligned}$$

(Here the first pair of brackets  $\langle \cdot, \cdot \rangle_{1,-1}$  is the pairing of  $\mathcal{H}_1(\Delta)$  and  $\mathcal{H}_{-1}(\Delta)$ ). In particular, the value  $\mathcal{E}_\lambda^\Delta \varphi_j^\Delta = \varphi_j^\Delta(\lambda)$  of the vector  $\varphi_j^\Delta$  at  $\lambda$  is defined by

$$\mathcal{E}_\lambda^\Delta \varphi_j^\Delta = \frac{1}{k_j^\Delta} \eta_j^\Delta(\lambda).$$

The Hilbert space  $\mathfrak{h}_\lambda(\Delta)$  is defined as the closure of the image of the evaluation operator

$$(39) \quad \mathfrak{h}_\lambda(\Delta) = \overline{\mathcal{E}_\lambda^\Delta \mathcal{H}_1(\Delta)},$$

where  $\mathcal{H}_1(\Delta)$  is the Hilbert space  $\mathcal{H}_1(F_\Delta)$  of regular vectors of the Hilbert space  $E_\Delta \mathcal{H}$  with respect to the rigging  $F_\Delta$  (see (24) and (28)).

**3.5.1. Natural isomorphisms  $U_{\Delta_2, \Delta_1}(\lambda)$ .** Let  $\lambda \in \Lambda(H, F)$  and let  $\Delta_1, \Delta_2 \in \mathcal{B}_\lambda$ . We define the gluing unitary operators  $U_{\Delta_2, \Delta_1}(\lambda)$  by the formula

$$(40) \quad U_{\Delta_2, \Delta_1}(\lambda) \mathcal{E}_{\lambda_1}^{\Delta_1} E_{\Delta_1} f := \mathcal{E}_{\lambda_2}^{\Delta_2} E_{\Delta_2} f,$$

where  $f \in \mathcal{H}_1(F)$ . Note that by (28) vectors  $E_{\Delta_j} f$ ,  $f \in \mathcal{H}_1(F)$ , exhaust the Hilbert space  $\mathcal{H}_1(\Delta_j)$ , and by (39) and (28), the Hilbert space  $\mathfrak{h}_\lambda(\Delta_j)$  is the completion of the linear manifold

$$\mathcal{E}_\lambda^{\Delta_j} E_{\Delta_j} \operatorname{ran}(F^*) = \mathcal{E}_\lambda^{\Delta_j} \mathcal{H}_1(\Delta_j).$$

It follows that it is sufficient to define  $U_{\Delta_2, \Delta_1}(\lambda)$  on this image.

**Proposition 3.12.** *The operator  $U_{\Delta_2, \Delta_1}(\lambda): \mathfrak{h}_\lambda(\Delta_1) \rightarrow \mathfrak{h}_\lambda(\Delta_2)$ , defined by formula (40), is unitary.*

*Proof.* What we have to show is that for any  $f \in \mathcal{H}_1(F)$  the norm of the vector  $\mathcal{E}_\lambda^\Delta E_\Delta f$  of the Hilbert space  $\mathfrak{h}_\lambda(\Delta)$  does not depend on the choice of a bounded open subset

$\Delta \subset \Lambda(H, F)$  as long as it contains  $\lambda$ . So, let  $\Delta$  be such a set and let  $(\beta_j^\Delta)$  be an  $\ell_2$ -sequence such that (see (26))

$$E_\Delta f = \sum_{j=1}^{\infty} \beta_j^\Delta \kappa_j^\Delta \varphi_j^\Delta.$$

By definition of the scalar product in  $\mathfrak{h}_\lambda(\Delta) \subset \ell_2$ , we have

$$\|\mathcal{E}_\lambda^\Delta E_\Delta f\|_{\mathfrak{h}_\lambda(\Delta)}^2 = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \bar{\beta}_i^\Delta \beta_j^\Delta \kappa_i^\Delta \kappa_j^\Delta \langle \mathcal{E}_\lambda^\Delta \varphi_i^\Delta, \mathcal{E}_\lambda^\Delta \varphi_j^\Delta \rangle,$$

where the double series converges absolutely (see the remark after formula [Az, (3.1)]).

Hence, using [Az, (5.5)] we obtain

$$\begin{aligned} \|\mathcal{E}_\lambda^\Delta E_\Delta f\|_{\mathfrak{h}_\lambda(\Delta)}^2 &= \frac{1}{\pi} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \bar{\beta}_i^\Delta \beta_j^\Delta \kappa_i^\Delta \kappa_j^\Delta \langle \varphi_i^\Delta, \text{Im } R_{\lambda+iy}(E_\Delta^{H_0} H) \varphi_j^\Delta \rangle_{1,-1} \\ (41) \quad &= \frac{1}{\pi} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \bar{\beta}_i^\Delta \beta_j^\Delta \kappa_i^\Delta \kappa_j^\Delta \lim_{y \rightarrow 0^+} \langle \varphi_i^\Delta, \text{Im } R_{\lambda+iy}(E_\Delta^{H_0} H) \varphi_j^\Delta \rangle \end{aligned}$$

Since  $\lambda \in \Lambda(H, F)$ , the trace-class valued matrix

$$\varphi_{\lambda+iy}^\Delta = \frac{1}{\pi} (\kappa_i^\Delta \kappa_j^\Delta \langle \varphi_i^\Delta, \text{Im } R_{\lambda+iy}(E_\Delta^{H_0} H) \varphi_j^\Delta \rangle)_{i,j=1}^\infty$$

continuously depends on  $y$  up to  $y = 0$  (see part II of § 1.1). So, the function  $\langle \beta, \varphi_{\lambda+iy}^\Delta \beta \rangle$  is also continuous up to  $y = 0$ . Therefore, we can interchange the summations and the limit operation in (41) to get

$$\begin{aligned} \|\mathcal{E}_\lambda^\Delta E_\Delta f\|_{\mathfrak{h}_\lambda(\Delta)}^2 &= \frac{1}{\pi} \lim_{y \rightarrow 0^+} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \bar{\beta}_i^\Delta \beta_j^\Delta \kappa_i^\Delta \kappa_j^\Delta \langle \varphi_i^\Delta, \text{Im } R_{\lambda+iy}(E_\Delta^{H_0} H) \varphi_j^\Delta \rangle \\ &= \frac{1}{\pi} \lim_{y \rightarrow 0^+} \langle E_\Delta f, \text{Im } R_{\lambda+iy}(E_\Delta^{H_0} H) E_\Delta f \rangle. \end{aligned}$$

The last expression does not depend on  $\Delta$  as long as it contains  $\lambda$ . □

Definition (40) of operators  $U_{\Delta_2, \Delta_1}(\lambda)$  implies that they satisfy the gluing property (14). Thus, to any compatible pair  $(H, F)$  and a number  $\lambda \in \Lambda(H, F)$  we can assign a sheaf of fiber Hilbert spaces

$$(42) \quad \mathfrak{h}_\lambda(H, F) = \{\mathfrak{h}_\lambda(\Delta), \Delta \in \mathcal{B}_\lambda\},$$

where the Hilbert spaces  $\mathfrak{h}_\lambda(\Delta)$  are defined by (39) and the gluing unitary isomorphisms  $U_{\Delta_2, \Delta_1}(\lambda)$  are defined by (40).

**3.6. Sheaf  $\mathcal{S}(H, F)$ .** By [Az, Corollary 3.1.8], for any bounded open set  $\Delta \subset \mathbb{R}$  the family of Hilbert spaces  $\{\mathfrak{h}_\lambda(\Delta), \lambda \in \Delta\}$  is measurable. It follows from the definition (40) of the gluing unitary operators  $U_{\Delta_2, \Delta_1}(\lambda)$  that for any sets  $\Delta_1, \Delta_2$  from  $\mathcal{B}$ , the family of operators  $\{U_{\Delta_2, \Delta_1}(\lambda), \lambda \in \Delta_1 \cap \Delta_2\}$  is also measurable. Thus, for any compatible pair  $(H, F)$  we have assigned a sheaf of Hilbert spaces over  $\Lambda(H, F)$ . We shall denote this sheaf of Hilbert spaces by  $\mathcal{S}(H, F)$ . We shall also use the notions and notation associated with a sheaf of Hilbert spaces, such as  $\mathcal{H}(\Delta)$ , etc, introduced in § 2.

**3.7. Evaluation operator  $\mathcal{E}_\lambda(H)$ .** Let  $\lambda \in \Lambda(H, F)$ . For any  $f \in \mathcal{H}_1(F)$  and for any bounded open subset  $\Delta$  of  $\Lambda(H, F)$ , such that  $\lambda \in \Delta$ , we have a vector

$$(43) \quad f_\Delta(\lambda) := \mathcal{E}_\lambda^\Delta E_\Delta f \in \mathfrak{h}_\lambda(\Delta).$$

It follows from (40) that the family

$$f(\lambda) := \{f_\Delta(\lambda) : \Delta \in \mathcal{B}_\lambda\}$$

correctly defines an element of the sheaf of fiber Hilbert spaces (42). Further, by definition (17), the formula

$$(44) \quad \mathcal{E}_\lambda(H)f = \{f_\Delta(\lambda) : \Delta \in \mathcal{B}_\lambda\}$$

combined with (40) correctly defines an operator

$$(45) \quad \mathcal{E}_\lambda(H) : \mathcal{H}_1(F) \rightarrow \mathfrak{h}_\lambda(H, F).$$

We often omit the argument  $H$  in  $\mathcal{E}_\lambda(H)$ , if there is no danger of confusion. According to (16), for any  $f, g \in \mathcal{H}_1(F)$  and any  $\Delta \in \mathcal{B}_\lambda$

$$(46) \quad \langle \mathcal{E}_\lambda f, \mathcal{E}_\lambda g \rangle_{\mathfrak{h}_\lambda(H, F)} = \langle \mathcal{E}_\lambda^\Delta E_\Delta f, \mathcal{E}_\lambda^\Delta E_\Delta g \rangle_{\mathfrak{h}_\lambda(\Delta)}.$$

**Proposition 3.13.** *The operator (45), defined by equality (44), is Hilbert-Schmidt.*

*Proof.* Let  $f_1, f_2, \dots$  be an orthonormal basis of  $\mathcal{H}_1(F)$ . For any  $\Delta \in \mathcal{B}_\lambda$ , we have

$$\|\mathcal{E}_\lambda\|_{\mathcal{L}_2(\mathcal{H}_1(F), \mathfrak{h}_\lambda)}^2 = \sum_{j=1}^{\infty} \|\mathcal{E}_\lambda f_j\|_{\mathfrak{h}_\lambda}^2 = \sum_{j=1}^{\infty} \|\mathcal{E}_\lambda^\Delta E_\Delta f_j\|_{\mathfrak{h}_\lambda(\Delta)}^2,$$

where the second equality follows from (46). By Lemma 3.7, the operator  $E_\Delta$ , considered as an operator  $\mathcal{H}_1(F) \rightarrow \mathcal{H}_1(\Delta)$ , is bounded, and the operator

$$\mathcal{E}_\lambda^\Delta : \mathcal{H}_1(\Delta) \rightarrow \mathfrak{h}_\lambda(\Delta)$$

is Hilbert-Schmidt (see part IV of § 1.1 and/or [Az, §3.1]). It follows that the composition  $\mathcal{E}_\lambda^\Delta E_\Delta$  is also Hilbert-Schmidt and therefore the sum above is finite. It follows that  $\mathcal{E}_\lambda$  is also Hilbert-Schmidt.  $\square$

We say that two vectors  $f$  and  $g$  from  $\mathcal{H}_1(F)$  are *equivalent at  $\lambda$* , if  $E_\Delta f = E_\Delta g$  for some  $\Delta \in \mathcal{B}_\lambda$ . Obviously, equivalence at  $\lambda$  is an equivalence relation. A jet of vectors at  $\lambda$  is an equivalence class with respect to this equivalence relation and the set of all jets of vectors at  $\lambda$  is a linear space. As can be seen from the definition of the operator  $\mathcal{E}_\lambda$ , it is in fact defined on the space of jets of vectors at  $\lambda$ .

**3.7.1. Operator  $\mathcal{E}$ .** For any bounded open subset  $\Delta$  of  $\Lambda(H, F)$  we define the operator

$$(47) \quad \mathcal{E}^\Delta : \mathcal{H}_1(\Delta) \rightarrow \mathcal{H}(\Delta), \quad \mathcal{E}^\Delta = \int_{\Delta}^{\oplus} \mathcal{E}_\lambda^\Delta d\lambda,$$

where  $\mathcal{E}_\lambda^\Delta$  is defined by (36).

**Lemma 3.14.** *The operator  $\mathcal{E}^\Delta$  defined by (47) and considered as an operator from  $E_\Delta \mathcal{H}$  to  $\mathcal{H}(\Delta)$  is bounded and unitary. Moreover, for any  $f_\Delta \in \mathcal{H}_1(\Delta)$ , and for a.e.  $\lambda \in \Delta$ .*

$$[\mathcal{E}^\Delta(Hf_\Delta)](\lambda) = \lambda \mathcal{E}_\lambda^\Delta(f_\Delta).$$

*Proof.* Comparing definitions (36), (37) and (38) with definitions (9), (5) and (4), we infer that the operator  $\mathcal{E}^\Delta$  is the evaluation operator (see parts IV-VI of §1.1) corresponding to self-adjoint operator  $HE_\Delta$  on the Hilbert space  $E_\Delta\mathcal{H}$  with Hilbert-Schmidt rigging  $F_\Delta = FE_\Delta$ . Therefore, the required assertion is a direct corollary of Theorem 1.3.  $\square$

Now we give definition of the evaluation operator

$$\mathcal{E}: \mathcal{H} \rightarrow \mathcal{S}(H, F).$$

We shall do it in two different ways, mentioned in subsection 2.2.1. It is not difficult to see that the family of operators

$$\{\mathcal{E}_\lambda \in \mathcal{L}_2(\mathcal{H}_1(F), \mathfrak{h}_\lambda(H, F)): \lambda \in \Lambda(H, F)\},$$

which are defined by (45), is measurable. Thus, we can define the operator

$$\mathcal{E} = \int_{\Lambda}^{\oplus} \mathcal{E}_\lambda d\lambda,$$

which acts from the dense subspace  $\mathcal{H}_1(F)$  of  $\mathcal{H}$  to the sheaf of Hilbert spaces  $\mathcal{S}(H, F)$ . Further, comparing definition (44) and (43) of  $\mathcal{E}_\lambda$  and definition (47) of  $\mathcal{E}^\Delta$  we infer that for any  $f \in \mathcal{H}_1(F)$

$$(48) \quad (\mathcal{E}f)_\Delta = \mathcal{E}^\Delta E_\Delta f,$$

where  $E_\Delta$  is considered here as an operator acting from  $\mathcal{H}_1(F)$  onto  $\mathcal{H}_1(\Delta)$  (see Lemma 3.7). Since  $\mathcal{H}_1(F)$  is naturally embedded into  $\mathcal{H}$ , we can consider  $\mathcal{E}$  as an operator from  $\mathcal{H}$  to  $\mathcal{S}(H, F)$ . The equality (48) gives a second equivalent definition of  $\mathcal{E}$ . Finally, the collection of mappings

$$\{\mathcal{E}_\lambda^\Delta: \mathcal{H}_1(\Delta) \rightarrow \mathfrak{h}_\lambda(\Delta) \mid \lambda \in \Lambda(H, F), \Delta \in \mathcal{B}_\lambda\},$$

defined by (36) and combined with gluing property (40) can also be considered as a definition of  $\mathcal{E}$ .

**Theorem 3.15.** *The operator  $\mathcal{E}$  defines a bounded operator from  $\mathcal{H}$  to the sheaf of Hilbert spaces  $\mathcal{S}(H, F)$ . The operator  $\mathcal{E}$  vanishes on singular (with respect to  $H$ ) subspace of  $\mathcal{H}$  and it is a unitary isomorphism of the absolutely continuous (with respect to  $H$ ) part of  $\mathcal{H}$  onto the sheaf  $\mathcal{S}(H, F)$ . This mapping diagonalizes the absolutely continuous part of the operator  $H$ , that is, for any  $f \in \text{dom}(H)$  and for a.e.  $\lambda \in \Lambda(H, F)$  the equality  $\mathcal{E}_\lambda(Hf) = \lambda \mathcal{E}_\lambda(f)$  holds.*

*Proof.* For any  $f \in \mathcal{H}_1(F)$ , it follows from (18), (48) and Theorem 1.3, that

$$(49) \quad \begin{aligned} \|\mathcal{E}f\|_{\mathcal{S}(H, F)} &= \sup_{\Delta \in \mathcal{B}} \|(\mathcal{E}f)_\Delta\|_{\mathcal{H}(\Delta)} = \sup_{\Delta \in \mathcal{B}} \|\mathcal{E}^\Delta E_\Delta f\|_{\mathcal{H}(\Delta)} \\ &= \sup_{\Delta \in \mathcal{B}} \|P^{(a)} E_\Delta f\|_{E_\Delta \mathcal{H}} = \|P^{(a)} f\|_{\mathcal{H}}. \end{aligned}$$

Since  $\mathcal{H}_1(F)$  is densely included in  $\mathcal{H}$ , it follows that  $\mathcal{E}$  can be continued to a bounded operator on  $\mathcal{H}$ , which is a partial isometry with initial space  $P^{(a)}\mathcal{H}$ . By definition (48) of  $\mathcal{E}$ , on the subspace  $E_\Delta\mathcal{H}$  the operator  $\mathcal{E}$  coincides with the unitary operator  $\mathcal{E}^\Delta: E_\Delta\mathcal{H} \rightarrow \mathcal{H}(\Delta)$ .

By Lemma 3.14, the unitary operator  $\mathcal{E}^\Delta: E_\Delta \mathcal{H} \rightarrow \mathcal{H}(\Delta)$  diagonalizes the self-adjoint operator  $E_\Delta H$ , that is,  $\mathcal{E}^\Delta(E_\Delta H f) = \int_\Delta^\oplus \lambda \mathcal{E}^\Delta(E_\Delta f) d\lambda$ . It follows from this and (48) that for any  $f \in \mathcal{H}$

$$\mathcal{E}(Hf) = \int_{\Lambda(H,F)}^\oplus \mathcal{E}_\lambda(Hf) d\lambda = \int_{\Lambda(H,F)}^\oplus \lambda \mathcal{E}_\lambda f d\lambda.$$

Since the image of  $\mathcal{E}^\Delta$  coincides with  $\mathcal{H}(\Delta)$ , it also follows from the last equality that the operator  $\mathcal{E}: \mathcal{H} \rightarrow \mathcal{S}(H, F)$  is onto.  $\square$

**Corollary 3.16.** *If  $h$  is a bounded measurable function on  $\mathbb{R}$  and  $f \in \mathcal{H}$ , then for a.e.  $\lambda \in \Lambda(H, F)$  we have  $\mathcal{E}_\lambda(h(H)f) = h(\lambda)\mathcal{E}_\lambda(f)$ .*

**3.8. Green operator  $R_{\lambda+i0}(H)$ .** If a self-adjoint operator  $H$  is compatible with a rigging  $F$ , then by Lemma 3.4 the operator  $FR_z(H)F^*$  is compact. By definition, the norm limit  $FR_{\lambda+i0}(H)F^*$  exists for all  $\lambda$  from the set of full Lebesgue measure  $\Lambda(H, F)$ . Thus,  $FR_{\lambda+i0}(H)F^*$  is a compact operator on the Hilbert space  $\mathcal{K}$  for all  $\lambda \in \Lambda(H, F)$ . Since the operator  $F$  (respectively,  $F^*$ ) can be seen as a natural isomorphism of Hilbert spaces  $\mathcal{H}_{-1}(F)$  and  $\mathcal{K}$  (respectively,  $\mathcal{K}$  and  $\mathcal{H}_1(F)$ ), the operator  $FR_{\lambda+i0}(H)F^*$  can be interpreted as a compact limit

$$(50) \quad R_{\lambda+i0}(H): \mathcal{H}_1 \rightarrow \mathcal{H}_{-1}$$

of the operator  $R_{\lambda+i0}(H): \mathcal{H}_1 \rightarrow \mathcal{H}_{-1}$  in the norm of  $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_{-1})$ . In this subsection we study properties of the operator (50).

We denote by  $R_{\lambda+iy}^\Delta(H)$  the operator  $E_\Delta R_{\lambda+iy}(H)$ . The operator  $R_{\lambda+iy}^\Delta(H)$  can be considered as an operator on the Hilbert space  $E_\Delta$  with Hilbert-Schmidt rigging  $F_\Delta$ . So, there exists the limit operator  $R_{\lambda+i0}^\Delta(H)$  which acts from  $\mathcal{H}_1(\Delta)$  to  $\mathcal{H}_{-1}(\Delta)$ , provided that  $\lambda \in \Lambda(H, F) \cap \Delta$ . The operator  $R_{\lambda+i0}^\Delta(H)$  can also be interpreted as an operator from  $\mathcal{H}_1(F)$  to  $\mathcal{H}_{-1}(F)$  as a composition

$$\mathcal{H}_{-1}(F) \hookleftarrow \mathcal{H}_{-1}(\Delta) \xleftarrow{R_{\lambda+i0}^\Delta(H)} \mathcal{H}_1(\Delta) \xleftarrow{E_\Delta} \mathcal{H}_1(F).$$

Further, the operator  $R_{\lambda+i0}^\Delta(H)$  can also be naturally defined as the norm limit of the operator  $R_{\lambda+iy}^\Delta(H): \mathcal{H}_1(F) \rightarrow \mathcal{H}_{-1}(F)$ . In the following lemma we show that these two interpretations are identical.

**Lemma 3.17.** *Let  $\lambda \in \Lambda(H, F)$ . For any  $\Delta \in \mathcal{B}_\lambda$  the limit  $R_{\lambda+i0}^\Delta(H)$  of the cut off resolvent  $R_{\lambda+iy}^\Delta(H)$  as  $y \rightarrow 0$  exists in the norm topology of  $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_{-1})$ . Further, the norm limit  $R_{\lambda+i0}^\Delta(H) \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_{-1})$  above coincides with the composition of the contraction  $E_\Delta: \mathcal{H}_1(F) \rightarrow \mathcal{H}_1(\Delta)$ , the norm limit  $R_{\lambda+i0}^\Delta$  of  $R_{\lambda+iy}^\Delta$  as an operator from  $\mathcal{H}_1(\Delta)$  to  $\mathcal{H}_{-1}(\Delta)$  and the inclusion operator  $\mathcal{H}_{-1}(\Delta) \hookrightarrow \mathcal{H}_{-1}(F)$ .*

*Proof.* For any  $f, g \in \mathcal{H}_1(F)$  we have

$$\begin{aligned} \langle f, R_{\lambda+iy}^\Delta(H)g \rangle_{\mathcal{H}_1(F), \mathcal{H}_{-1}(F)} &= \langle f, R_{\lambda+iy}^\Delta(H)g \rangle_{\mathcal{H}} \\ (51) \quad &= \langle E_\Delta f, R_{\lambda+iy}^\Delta(H)E_\Delta g \rangle_{E_\Delta \mathcal{H}} \\ &= \langle E_\Delta f, R_{\lambda+iy}^\Delta(H)E_\Delta g \rangle_{\mathcal{H}_1(\Delta), \mathcal{H}_{-1}(\Delta)}, \end{aligned}$$



where according to Lemma 3.6 in the last equality we can consider vectors  $E_\Delta f$  and  $E_\Delta g$  as elements of the Hilbert space  $\mathcal{H}_1(\Delta)$ . The right hand side of (51) has a limit as  $y \rightarrow 0^+$ , since according to (34) and (35) we have  $\lambda \in \Lambda(HE_\Delta, F_\Delta)$ . The left hand side of (51) has a limit, since for  $\lambda \in \Lambda(H, F)$  the limit  $\lim_{y \rightarrow 0} R_{\lambda+iy}(H)$  exists in the norm of  $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_{-1})$  and the limit  $\lim_{y \rightarrow 0} R_{\lambda+iy}^{\mathbb{R} \setminus \Delta}(H)$  exists even in the norm of  $\mathcal{B}(\mathcal{H})$ . Taking these limits we obtain

$$\langle f, R_{\lambda+i0}^\Delta(H)g \rangle_{\mathcal{H}_1(F), \mathcal{H}_{-1}(F)} = \langle E_\Delta f, R_{\lambda+i0}^\Delta(H)E_\Delta g \rangle_{\mathcal{H}_1(\Delta), \mathcal{H}_{-1}(\Delta)}.$$

This equality completes the proof.  $\square$

Next we show that the resolvent  $R_{\lambda+i0}(H)$  can be treated as the limit of the operator  $R_{\lambda+i0}^\Delta(H)$  as  $\Delta \rightarrow \mathbb{R}$ .

**Proposition 3.18.** *If  $H$  is a self-adjoint operator on a Hilbert space  $\mathcal{H}$  compatible with a rigging  $F$  and if  $\lambda \in \Lambda(H, F)$ , then there holds the equality*

$$R_{\lambda+i0}(H) = \lim_{\Delta \nearrow \mathbb{R}} R_{\lambda+i0}^\Delta(H),$$

where the limit is taken in norm topology of  $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_{-1})$ .

Proof of this proposition is the same as that of Lemma 3.4 and therefore is omitted.

The operator  $\text{Im } R_{\lambda+i0}(H): \mathcal{H}_1(F) \rightarrow \mathcal{H}_{-1}(F)$  is compact as a difference of two compact operators. We now study some additional properties of this operator.

**Lemma 3.19.** *If  $H$  is a self-adjoint operator on a rigged Hilbert space  $(\mathcal{H}, F)$  which is compatible with a rigging  $F$ , if  $\lambda \in \Lambda(H, F)$  and if  $\Delta \in \mathcal{B}_\lambda$ , then for any  $f, g \in \mathcal{H}_1(F)$*

$$(52) \quad \langle f, \text{Im } R_{\lambda+i0}(H)g \rangle_{1,-1} = \langle f, \text{Im } R_{\lambda+i0}^\Delta(H)g \rangle_{1,-1}.$$

*Proof.* It follows from the first equality of (51) that the right hand side of (52) does not depend on  $\Delta \in \mathcal{B}_\lambda$ . This observation, combined with Proposition 3.18, completes the proof.  $\square$

**Proposition 3.20.** *If  $H$  is a self-adjoint operator on a Hilbert space  $\mathcal{H}$  and is compatible with a rigging  $F$  and if  $\lambda \in \Lambda(H, F)$ , then*

$$\frac{1}{\pi} \text{Im } R_{\lambda+i0}(H) = \mathcal{E}_\lambda^\diamond(H) \mathcal{E}_\lambda(H)$$

as the equality in  $\mathcal{B}(\mathcal{H}_1(F), \mathcal{H}_{-1}(F))$ . In particular, the operator  $\text{Im } R_{\lambda+i0}(H)$  is trace class.

*Proof.* It is enough to show that for any  $f, g \in \mathcal{H}_1(F)$  the equality

$$\frac{1}{\pi} \langle f, \text{Im } R_{\lambda+i0}(H)g \rangle_{1,-1} = \langle f, \mathcal{E}_\lambda^\diamond \mathcal{E}_\lambda g \rangle_{1,-1}$$

holds. For any  $\Delta \in \mathcal{B}_\lambda$ , by (46) and [Az, (5.5)], the right hand side of this equality is equal to

$$\begin{aligned} \langle \mathcal{E}_\lambda f, \mathcal{E}_\lambda g \rangle_{\mathfrak{h}_\lambda} &= \langle \mathcal{E}_\lambda^\Delta E_\Delta f, \mathcal{E}_\lambda^\Delta E_\Delta g \rangle_{\mathfrak{h}_\lambda(\Delta)} \\ &= \frac{1}{\pi} \langle E_\Delta f, \text{Im } R_{\lambda+i0}^\Delta(H)E_\Delta g \rangle_{\mathcal{H}_1(\Delta), \mathcal{H}_{-1}(\Delta)}, \end{aligned}$$

where  $\text{Im } R_{\lambda+i0}^\Delta(H)$  is considered as an operator from  $\mathcal{H}_1(\Delta)$  to  $\mathcal{H}_{-1}(\Delta)$ . Thus, it follows from (51) that

$$\langle f, \mathcal{E}_\lambda^\diamond \mathcal{E}_\lambda g \rangle_{1,-1} = \frac{1}{\pi} \langle f, \text{Im } R_{\lambda+i0}^\Delta(H) g \rangle_{1,-1},$$

where now  $\text{Im } R_{\lambda+i0}^\Delta(H): \mathcal{H}_1(F) \rightarrow \mathcal{H}_{-1}(F)$ . Combining this with Lemma 3.19 completes the proof.  $\square$

#### 4. WAVE MATRIX

So far we have considered a single self-adjoint operator  $H$  on a Hilbert space  $\mathcal{H}$  with a fixed rigging operator  $F$ , and for this reason we did not use sub-indices. From now on we are going to consider not only the operator  $H$  but also its perturbations. We denote by  $H_0$  an “initial” self-adjoint operator, and by  $H_1 = H_0 + V$  we denote its perturbation by a self-adjoint operator  $V$ . We assume that the operator  $V$  admits a decomposition

$$(53) \quad V = F^* J F,$$

where  $J$  is a bounded operator on  $\mathcal{K}$ . Existence of such a factorization allows to treat  $V$  as a bounded operator (see e.g. discussion in introduction)

$$(54) \quad V: \mathcal{H}_{-1}(F) \rightarrow \mathcal{H}_1(F).$$

**4.1. Operators  $\mathfrak{a}_\pm(\lambda; H_1, H_0)$ .** We shall assume that  $H_0$  and  $H_1$  are two self-adjoint operators on a rigged Hilbert space  $(\mathcal{H}, F)$ , compatible with the rigging  $F$  and such that the difference  $V = H_1 - H_0$  admits decomposition (53). This means that  $V$  can be considered as a bounded operator (54) and we shall do this without further reference. Whether  $V$  is considered as an operator  $\mathcal{H} \rightarrow \mathcal{H}$  or as an operator (54) will be clear from the context.

With the preparations given in previous sections, for any real number  $\lambda$ , which belongs to the set  $\Lambda(H_0, F) \cap \Lambda(H_1, F)$ , we can now define operators

$$\mathfrak{a}_\pm(\lambda; H_1, H_0): \mathcal{H}_1(F) \rightarrow \mathcal{H}_{-1}(F),$$

which are analogues of the forms [Y, (2.7.4)]. We have (cf. e.g. [Y, (2.7.10)], see also [Az, (5.3)])

$$(55) \quad \begin{aligned} \frac{y}{\pi} R_{\lambda \mp iy}(H_1) R_{\lambda \pm iy}(H_0) &= \frac{1}{\pi} \text{Im } R_{\lambda+iy}(H_1) \left[ 1 + V R_{\lambda \pm iy}(H_0) \right] \\ &= \left[ 1 - R_{\lambda \mp iy}(H_1) V \right] \cdot \frac{1}{\pi} \text{Im } R_{\lambda+iy}(H_0). \end{aligned}$$

Since  $\lambda \in \Lambda(H_0, F) \cap \Lambda(H_1, F)$ , by Definition 3.9 the limits

$$(56) \quad \begin{aligned} \mathfrak{a}_\pm(\lambda; H_1, H_0) &:= \frac{1}{\pi} \text{Im } R_{\lambda+i0}(H_1) \left[ 1 + V R_{\lambda \pm i0}(H_0) \right] \\ &= \left[ 1 - R_{\lambda \mp i0}(H_1) V \right] \cdot \frac{1}{\pi} \text{Im } R_{\lambda+i0}(H_0) \end{aligned}$$

exist in norm topology of the space  $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_{-1})$ . Since the resolvent operator  $R_{\lambda+iy}(H)$  is compact as an operator from  $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_{-1})$ , it follows from the definition of the operators  $\mathfrak{a}_\pm(\lambda; H_1, H_0)$ , that they are also compact.

**4.2. Wave matrix  $w_{\pm}(\lambda; H_1, H_0)$ .** Let  $\lambda \in \Lambda(H_0, F) \cap \Lambda(H_1, F)$ . Following (12) (see also [Az, Definition 5.2.1]), we define the wave matrix  $w_{\pm}(\lambda; H_1, H_0)$  as a form

$$w_{\pm}(\lambda; H_1, H_0): \mathfrak{h}_{\lambda}(H_1) \times \mathfrak{h}_{\lambda}(H_0) \rightarrow \mathbb{C},$$

by the formula

$$(57) \quad w_{\pm}(\lambda; H_1, H_0) (\mathcal{E}_{\lambda}(H_1)f, \mathcal{E}_{\lambda}(H_0)g) = \langle f, \mathfrak{a}_{\pm}(\lambda; H_1, H_0)g \rangle_{1,-1},$$

where  $f, g \in \mathcal{H}_1(F)$ . This definition is exactly the same as the definition of the wave matrix for the case of trace-class perturbations  $V$ , considered in [Az, §5.2]; the only difference being the treatment of the fiber Hilbert space  $\mathfrak{h}_{\lambda}$  as a sheaf of Hilbert spaces (42). The wave matrix  $w_{\pm}(\lambda; H_1, H_0)$  is correctly defined by equality (57) in the sense that if  $f'$  and  $g'$  is another pair of vectors from  $\mathcal{H}_1(F)$  such that  $\mathcal{E}_{\lambda}(H_1)f = \mathcal{E}_{\lambda}(H_1)f'$  and  $\mathcal{E}_{\lambda}(H_0)g = \mathcal{E}_{\lambda}(H_0)g'$ , then

$$w_{\pm}(\lambda; H_1, H_0) (\mathcal{E}_{\lambda}(H_1)f, \mathcal{E}_{\lambda}(H_0)g) = w_{\pm}(\lambda; H_1, H_0) (\mathcal{E}_{\lambda}(H_1)f', \mathcal{E}_{\lambda}(H_0)g'),$$

as it follows from Proposition 3.20 and definition (56) of  $\mathfrak{a}_{\pm}(\lambda; H_1, H_0)$ .

Proof of the following proposition follows verbatim the proof of [Az, Proposition 5.2.2] (the idea of which was taken in its turn from [Y, §5.2]), with reference to Proposition 3.20 instead of [Az, (5.5)].

**Proposition 4.1.** *Let  $H_0$  and  $H_1$  be two self-adjoint operators on a rigged Hilbert space  $(\mathcal{H}, F)$  which are compatible with the rigging  $F$ , such that  $V = H_1 - H_0$  admits decomposition (53). For any  $\lambda \in \Lambda(H_1, F) \cap \Lambda(H_0, F)$  the form  $w_{\pm}(\lambda; H_1, H_0)$  is well-defined, and it is bounded with norm  $\leq 1$ .*

*Proof.* That  $w_{\pm}(\lambda; H_1, H_0)$  is well-defined has already been shown.

Further, by Schwarz inequality, for any  $f, g \in \mathcal{H}_1(F) \subset \mathcal{H}$

$$(58) \quad \begin{aligned} & \frac{y}{\pi} \left| \langle f, R_{\lambda-iy}(H_1)R_{\lambda+iy}(H_0)g \rangle_{\mathcal{H}} \right| \\ & \leq \frac{y}{\pi} \|R_{\lambda+iy}(H_1)f\|_{\mathcal{H}} \|R_{\lambda+iy}(H_0)g\|_{\mathcal{H}} \\ & = \frac{1}{\pi} \left| \langle f, \text{Im } R_{\lambda+iy}(H_1)f \rangle_{\mathcal{H}} \right|^{1/2} \cdot \left| \langle g, \text{Im } R_{\lambda+iy}(H_0)g \rangle_{\mathcal{H}} \right|^{1/2}. \end{aligned}$$

In this inequality we take the limit  $y \rightarrow 0^+$  to get, using Proposition 3.20 and (20),

$$\left| \langle f, \mathfrak{a}_{\pm}(\lambda; H_1, H_0)g \rangle_{1,-1} \right| \leq \|\mathcal{E}_{\lambda}(H_1)f\|_{\mathfrak{h}_{\lambda}(H_1)} \cdot \|\mathcal{E}_{\lambda}(H_0)g\|_{\mathfrak{h}_{\lambda}(H_0)}.$$

It follows that the wave matrix is bounded with bound less or equal to 1.  $\square$

Henceforth we shall identify the form  $w_{\pm}(\lambda; H_1, H_0)$  with the corresponding operator

$$(59) \quad w_{\pm}(\lambda; H_1, H_0): \mathfrak{h}_{\lambda}(H_0) \rightarrow \mathfrak{h}_{\lambda}(H_1).$$

It follows directly from the definition of the wave matrix (57) and Proposition 3.20 that

$$(60) \quad w_{\pm}(\lambda; H_0, H_0) = 1_{\mathfrak{h}_{\lambda}(H_0)}.$$

**4.3. Multiplicative property of the wave matrix.** In this subsection we prove the multiplicative property of the wave matrix. As is known (cf. e.g. [Y, 2.7.3]), proof of the multiplicative property is one of the key points of the stationary approach to scattering theory.

**Lemma 4.2.** *Let  $H_0, H_1, H_2$  be three self-adjoint operators on a rigged Hilbert space  $(\mathcal{H}, F)$ , which are compatible with the rigging  $F$  and such that the operators  $V_1 = H_1 - H_0$  and  $V_2 = H_2 - H_1$  admit decomposition (53). If  $\lambda \in \Lambda(H_0, F) \cap \Lambda(H_1, F) \cap \Lambda(H_2, F)$ , then the equality*

$$(61) \quad \mathbf{a}_\pm(\lambda; H_2, H_0) = (1 - R_{\lambda \mp i0}(H_2)V_2) \frac{1}{\pi} \operatorname{Im} R_{\lambda+i0}(H_1)(1 + V_1 R_{\lambda \pm i0}(H_0))$$

*holds, where both sides are compact operators from  $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_{-1})$ .*

*Proof.* Let  $y > 0$ . We have

$$(62) \quad \begin{aligned} (E) &:= (1 - R_{\lambda \mp iy}(H_2)V_2) \frac{1}{\pi} \operatorname{Im} R_{\lambda+iy}(H_1)(1 + V_1 R_{\lambda \pm iy}(H_0)) \\ &= \frac{y}{\pi} (1 - R_{\lambda \mp iy}(H_2)V_2) R_{\lambda \mp iy}(H_1) R_{\lambda \pm iy}(H_1) (1 + V_1 R_{\lambda \pm iy}(H_0)). \end{aligned}$$

Applying the second resolvent identity to this equality twice yields

$$(63) \quad (E) = \frac{y}{\pi} R_{\lambda \mp iy}(H_2) R_{\lambda \pm iy}(H_0).$$

Since for  $\lambda \in \Lambda(H_j, F)$  the limits  $R_{\lambda \pm i0}(H_j)$  of the resolvents exist in the norm topology of  $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_{-1})$ , it follows that for all  $\lambda \in \Lambda(H_0, F) \cap \Lambda(H_1, F) \cap \Lambda(H_2, F)$  the limit of the left hand side of (62) exists in norm as  $y \rightarrow 0^+$ . By (55) and (56), the norm limit of (63) as  $y \rightarrow 0^+$  exists in the norm of the space  $\mathcal{B}(\mathcal{H}_1(F), \mathcal{H}_{-1}(F))$  and is equal to  $\mathbf{a}_\pm(\lambda; H_2, H_0)$ .  $\square$

Let  $\lambda \in \Lambda(H, F)$  and let  $\Delta$  be any fixed open bounded set containing  $\lambda$ . We can consider a compatible pair  $(E_\Delta H, F_\Delta)$  with Hilbert-Schmidt rigging  $F_\Delta$ . By [Az, (2.20), 2.16(viii)] and Lemma 3.1.6], for such riggings there exists a sequence of vectors  $b_j^\Delta(\lambda + i0) \in \mathcal{H}_1(\Delta)$ ,  $j$  is an index of non-zero type (see [Az, §2.10]), such that the sequence of vectors

$$e_j^\Delta(\lambda + i0) := \mathcal{E}_\lambda^\Delta(H) b_j^\Delta(\lambda + i0) \in \mathfrak{h}_\lambda(\Delta), \quad j \in \mathcal{Z}_\lambda,$$

is an orthonormal basis of the Hilbert space  $\mathfrak{h}_\lambda(\Delta)$ , where  $\mathcal{Z}_\lambda$  is the set of indices of non-zero type. Since, by Lemma 3.7,  $\mathcal{H}_1(\Delta) = E_\Delta \mathcal{H}_1(F)$ , there exist vectors  $b_j(\lambda + i0) \in \mathcal{H}_1(F)$ , such that

$$b_j^\Delta(\lambda + i0) = E_\Delta b_j(\lambda + i0).$$

Thus, according to (46), the set of vectors

$$\{b_j(\lambda + i0) \in \mathcal{H}_1(F) : j \in \mathcal{Z}_\lambda\}$$

is such that the sequence

$$(64) \quad (\mathcal{E}_\lambda(H) b_j(\lambda + i0))_{j \in \mathcal{Z}_\lambda} \text{ is an orthonormal basis of } \mathfrak{h}_\lambda(H, F)$$

(see (42) for definition of the Hilbert space  $\mathfrak{h}_\lambda(H, F)$ ). It is the only property of vectors  $b_j(\lambda + i0)$  which is used in the proof of the following lemma.

**Lemma 4.3.** *Let  $H$  be a self-adjoint operator compatible with rigging  $F$  and let  $\lambda \in \Lambda(H, F)$ . For any  $f, g \in \mathcal{H}_1(F)$  the equality*

$$\begin{aligned} \frac{1}{\pi} \langle f, \operatorname{Im} R_{\lambda+i0}(H)g \rangle_{1,-1} \\ = \frac{1}{\pi^2} \sum_{j \in \mathcal{Z}_\lambda} \langle f, \operatorname{Im} R_{\lambda+i0}(H)b_j(\lambda+i0) \rangle_{1,-1} \langle \operatorname{Im} R_{\lambda+i0}(H)b_j(\lambda+i0), g \rangle_{-1,1} \end{aligned}$$

holds, where  $(b_j(\lambda+i0))_{j \in \mathcal{Z}_\lambda}$  is a sequence of vectors from  $\mathcal{H}_1(F)$ , constructed above, with the property (64).

*Proof.* Using Proposition 3.20 and (20), we infer that the right hand side of the equality to be proved is equal to

$$\begin{aligned} (E) &:= \sum_{j \in \mathcal{Z}_\lambda} \langle f, \mathcal{E}_\lambda^\diamond \mathcal{E}_\lambda b_j(\lambda+i0) \rangle_{1,-1} \langle \mathcal{E}_\lambda^\diamond \mathcal{E}_\lambda b_j(\lambda+i0), g \rangle_{-1,1} \\ &= \sum_{j \in \mathcal{Z}_\lambda} \langle \mathcal{E}_\lambda f, \mathcal{E}_\lambda b_j(\lambda+i0) \rangle_{\mathfrak{h}_\lambda} \langle \mathcal{E}_\lambda b_j(\lambda+i0), \mathcal{E}_\lambda g \rangle_{\mathfrak{h}_\lambda}. \end{aligned}$$

It follows from this and (64) that

$$(E) = \langle \mathcal{E}_\lambda f, \mathcal{E}_\lambda g \rangle_{\mathfrak{h}_\lambda} = \langle f, \mathcal{E}_\lambda^\diamond \mathcal{E}_\lambda g \rangle_{1,-1}.$$

Now, another application of Proposition 3.20 completes the proof.  $\square$

**Theorem 4.4.** *Let  $H_0, H_1, H_2$  be three self-adjoint operators compatible with rigging  $F$  on  $\mathcal{H}$ , which satisfy conditions of Lemma 4.2. If  $\lambda \in \Lambda(H_0, F) \cap \Lambda(H_1, F) \cap \Lambda(H_2, F)$ , then*

$$w_\pm(\lambda; H_2, H_0) = w_\pm(\lambda; H_2, H_1)w_\pm(\lambda; H_1, H_0).$$

*Proof.* Let  $f, g \in \mathcal{H}_1(F)$ . Since the linear manifold  $\mathcal{E}_\lambda(H)\mathcal{H}_1(F)$  is dense in  $\mathfrak{h}_\lambda(H)$ , it is enough to show that

$$\begin{aligned} (E) &:= \langle \mathcal{E}_\lambda(H_2)f, w_\pm(\lambda; H_2, H_1)w_\pm(\lambda; H_1, H_0)\mathcal{E}_\lambda(H_0)g \rangle \\ &= \langle \mathcal{E}_\lambda(H_2)f, w_\pm(\lambda; H_2, H_0)\mathcal{E}_\lambda(H_0)g \rangle. \end{aligned}$$

Let vectors  $b_j(\lambda+i0) \in \mathcal{H}_1(F)$ ,  $j \in \mathcal{Z}_\lambda$ , be as in Lemma 4.3 for the operator  $H_1$ . It follows from definition of  $(E)$  just given and (64) that

$$\begin{aligned} (E) &= \sum_{j \in \mathcal{Z}_\lambda} \langle \mathcal{E}_\lambda(H_2)f, w_\pm(\lambda; H_2, H_1)\mathcal{E}_\lambda(H_1)b_j(\lambda+i0) \rangle \\ &\quad \cdot \langle \mathcal{E}_\lambda(H_1)b_j(\lambda+i0), w_\pm(\lambda; H_1, H_0)\mathcal{E}_\lambda(H_0)g \rangle. \end{aligned}$$

Combining this equality with definition (57) of  $w_\pm(\lambda; H_1, H_0)$  gives

$$(E) = \sum_{j \in \mathcal{Z}_\lambda} \langle f, \mathfrak{a}_\pm(\lambda; H_2, H_1)b_j(\lambda+i0) \rangle_{1,-1} \langle b_j(\lambda+i0), \mathfrak{a}_\pm(\lambda; H_1, H_0)\mathcal{E}_\lambda(H_0)g \rangle_{1,-1}.$$

Combining this equality with formulas (56) for  $\mathbf{a}_\pm$  yields

$$(E) = \frac{1}{\pi^2} \sum_{j \in \mathcal{Z}_\lambda} \left\langle f, \left[ 1 - R_{\lambda \mp i0}(H_2)V \right] \operatorname{Im} R_{\lambda+i0}(H_1) b_j(\lambda + i0) \right\rangle_{1,-1} \\ \left\langle b_j(\lambda + i0), \operatorname{Im} R_{\lambda+i0}(H_1) \left[ 1 + V R_{\lambda \pm i0}(H_0) \right] g \right\rangle_{1,-1}.$$

By Lemma 4.3, it follows that

$$(E) = \frac{1}{\pi} \left\langle f, \left[ 1 - R_{\lambda \mp i0}(H_2)V \right] \operatorname{Im} R_{\lambda+i0}(H_1) \left[ 1 + V R_{\lambda \pm i0}(H_0) \right] g \right\rangle_{1,-1}.$$

Now, Lemma 4.2 implies that

$$(E) = \langle f, \mathbf{a}_\pm(\lambda; H_2, H_0)g \rangle_{1,-1} = \langle \mathcal{E}_\lambda(H_2)f, w_\pm(\lambda; H_2, H_0)\mathcal{E}_\lambda(H_0)g \rangle.$$

Proof is complete.  $\square$

**Corollary 4.5.** *For any two operators  $H_0, H_1$  which satisfy conditions of Proposition 4.1 and for any  $\lambda \in \Lambda(H_0, F) \cap \Lambda(H_1, F)$ , the wave matrix (59) is a unitary operator. Moreover,  $w_\pm^*(\lambda; H_1, H_0) = w_\pm(\lambda; H_0, H_1)$ .*

Proof of this corollary follows verbatim the proof of [Az, Corollary 5.3.8], but one has to use (60) and Proposition 4.1 instead of [Az, Propositions 5.2.3 and 5.2.2].

**Remark 4.6.** The argument of this proof works in the case of Hilbert-Schmidt rigging  $F$ , and thus it simplifies proof the multiplicative property of the wave matrix given in [Az].

**4.4. Wave operator.** Let  $H_0, H_1$  be operators which satisfy conditions of Proposition 4.1. Clearly, the operator-valued function  $\Lambda(H_0, F) \cap \Lambda(H_1, F) \ni \lambda \mapsto w_\pm(\lambda; H_1, H_0)$  is measurable. Therefore, as in [Az], we can define the wave operator as an operator

$$W_\pm(H_1, H_0): \mathcal{S}(H_0, F) \rightarrow \mathcal{S}(H_1, F)$$

by the formula

$$(65) \quad W_\pm(H_1, H_0) = \int_{\Lambda(H_0, F) \cap \Lambda(H_1, F)}^\oplus w_\pm(\lambda; H_1, H_0) d\lambda.$$

Since, by Theorem 3.15, the Hilbert space  $\mathcal{S}(H, F)$  is naturally isomorphic to the absolutely continuous subspace  $\mathcal{H}^{(a)}(H)$  of the self-adjoint operator  $H$ , it follows that the wave operator  $W_\pm(H_1, H_0)$ , thus defined, can also be considered as an operator

$$(66) \quad W_\pm(H_1, H_0): \mathcal{H}^{(a)}(H_0) \rightarrow \mathcal{H}^{(a)}(H_1).$$

As an immediate consequence of the definition (65), Theorem 4.4 and Corollary 4.5, we obtain the following theorem.

**Theorem 4.7.** *Let  $H_0, H_1, H_2$  be three self-adjoint operators which satisfy conditions of Lemma 4.2. The wave operator (66) defined by (65) have the following properties:*

- (i) *The operator (66) is unitary.*
- (ii)  $W_\pm(H_2, H_0) = W_\pm(H_2, H_1)W_\pm(H_1, H_0)$ .
- (iii)  $W_\pm^*(H_1, H_0) = W_\pm(H_0, H_1)$ .

(iv) *The operator  $W_{\pm}(H_0, H_0)$  is the identity operator on  $\mathcal{H}^{(a)}(H_0)$ .*

If we define the operator (66) to be zero on the singular subspace  $\mathcal{H}^{(s)}(H_0)$ , then the part (iv) of this theorem becomes  $W_{\pm}(H_0, H_0) = P^{(a)}(H_0)$ . It follows from (ii) with  $H_2 = H_0$ , (iii) and (iv) that

$$W_{\pm}(H_1, H_0) = W_{\pm}(H_1, H_0)P^{(a)}(H_0) = P^{(a)}(H_1)W_{\pm}(H_1, H_0).$$

**Theorem 4.8.** *For any bounded measurable function  $h$  on  $\mathbb{R}$  and any two operators self-adjoint operators  $H_0, H_1$  which satisfy conditions of Proposition 4.1, we have*

$$h(H_1)W_{\pm}(H_1, H_0) = W_{\pm}(H_1, H_0)h(H_0).$$

Also,

$$H_1W_{\pm}(H_1, H_0) = W_{\pm}(H_1, H_0)H_0.$$

*Proof.* This immediately follows from the definition (65) of the wave operator, Theorem 3.15 and Corollary 3.16.  $\square$

Hence, we have the following corollary (generalized Kato-Rosenblum theorem).

**Corollary 4.9.** *If  $H_0, H_1$  are two self-adjoint operators which satisfy conditions of Proposition 4.1, then restrictions of operators  $H_0$  and  $H_1$  to their absolutely continuous subspaces  $\mathcal{H}^{(a)}(H_0)$  and  $\mathcal{H}^{(a)}(H_1)$  respectively are unitarily equivalent.*

A natural question is whether definition (65) coincides with classical time-dependent definition of the wave matrix. The answer is positive: inspection of the proof of [Az, Theorem 6.1.4] (which itself is an adjustment of appropriate proofs from [Y] to our setting) shows that the following theorem holds.

**Theorem 4.10.** *If  $H_0$  and  $H_1$  are two self-adjoint operators which satisfy conditions of Proposition 4.1, then the strong operator limit*

$$\lim_{t \rightarrow \pm\infty} e^{itH_1} e^{-itH_0} P_0^{(a)}$$

*exists and coincides with  $\mathcal{E}^*(H_1)W_{\pm}(H_1, H_0)\mathcal{E}(H_0)$ .*

Proof of this theorem is the same as that of [Az, Theorem 6.1.4] with only one change: [Az, Lemma 6.1.1] should be replaced by the following lemma: if  $g \in \mathcal{H}$  is such that  $\|\mathcal{E}_{\lambda}(H_0)g\|_{\mathfrak{h}_{\lambda}} \leq N$  for a.e.  $\lambda \in \Lambda(H_0, F)$  and  $\mathcal{E}_{\lambda}(H_0)g = 0$  for a.e.  $\lambda \in \Lambda(H_0, F) \setminus \Delta$ , where  $\Delta$  is a bounded open set, then

$$\int_{-\infty}^{\infty} \left\| F e^{-itH_0} P_0^{(a)} g \right\|^2 dt \leq 2\pi N^2 \|F E_{\Delta}^{H_0}\|_2^2.$$

This lemma itself follows from [Az, Lemma 6.1.1] applied to the self-adjoint operator  $H_0 E_{\Delta}^{H_0}$  acting on the Hilbert space  $E_{\Delta}^{H_0} \mathcal{H}$  with Hilbert-Schmidt rigging  $F_{\Delta} = F E_{\Delta}^{H_0}$ . The set of vectors  $g$  which satisfy the premise of this lemma is dense in  $\mathcal{H}$ , and this is what is used in the proof of [Az, Theorem 6.1.4].

## 5. SCATTERING MATRIX

As in [Az, Section 7], given a number  $\lambda \in \Lambda(H_0, F) \cap \Lambda(H_1, F)$ , we define the scattering matrix  $S(\lambda; H_1, H_0)$  as an operator on the Hilbert space  $\mathfrak{h}_\lambda(H_0)$  by the formula

$$(67) \quad S(\lambda; H_1, H_0) = w_+^*(\lambda; H_1, H_0)w_-(\lambda; H_1, H_0).$$

Just as in [Az, Section 7], we list here some properties of the scattering matrix, which immediately follow from this definition and the properties of the wave matrix already established.

**Theorem 5.1.** *Let  $H_0, H_1, H_2$  be three self-adjoint operators compatible with rigging  $F$  on  $\mathcal{H}$ , which satisfy conditions of Lemma 4.2. Let  $\lambda \in \Lambda(H_0, F) \cap \Lambda(H_1, F) \cap \Lambda(H_2, F)$ . The scattering matrix (67) possesses the following properties.*

- (i) *The scattering matrix (67) is a unitary operator.*
- (ii)  $S(\lambda; H_2, H_0) = w_+^*(\lambda; H_1, H_0)S(\lambda; H_2, H_1)w_-(\lambda; H_1, H_0).$
- (iii)  $S(\lambda; H_2, H_0) = w_+^*(\lambda; H_1, H_0)S(\lambda; H_2, H_1)w_+(\lambda; H_1, H_0)S(\lambda; H_1, H_0).$

Another important property of the scattering matrix is the stationary formula.

**Theorem 5.2.** *Let  $H_0$  and  $H_1$  be two self-adjoint operators which satisfy conditions of Proposition 4.1. If  $\lambda \in \Lambda(H_0, F) \cap \Lambda(H_1, F)$ , then*

$$S(\lambda; H_1, H_0) = 1_{\mathfrak{h}_\lambda} - 2\pi i \mathcal{E}_\lambda(H_0)V(1 + R_{\lambda+i0}(H_0)V)^{-1}\mathcal{E}_\lambda^\diamond(H_0),$$

where  $\mathcal{E}_\lambda^\diamond(H_0)$  is defined by (20).

Recall that in this theorem  $V$  is understood as a bounded operator acting from  $\mathcal{H}_{-1}(F)$  to  $\mathcal{H}_1(F)$  and  $R_{\lambda+i0}(H_0)$  is understood as a compact operator acting from  $\mathcal{H}_1(F)$  to  $\mathcal{H}_{-1}(F)$ . Proof of this stationary formula follows verbatim the proof of [Az, Theorem 7.2.2] and therefore is omitted. Further, as a by-product of the proof of this theorem, as in [Az] one obtains the following formulas for the wave matrices

$$w_\pm(\lambda; H_1, H_0)\mathcal{E}_\lambda(H_0) = \mathcal{E}_\lambda(H_1)[1 + VR_{\lambda\pm i0}(H_0)].$$

**Corollary 5.3.** *If  $H_0, H_1$  are two self-adjoint operators which satisfy conditions of Proposition 4.1, and  $\lambda \in \Lambda(H_0, F) \cap \Lambda(H_1, F)$ , then  $S(\lambda; H_1, H_0) \in 1 + \mathcal{L}_1(\mathfrak{h}_\lambda(H_0))$ .*

*Proof.* This follows from Theorem 5.2 and the fact that, by Proposition 3.13, the evaluation operator  $\mathcal{E}_\lambda(H_0): \mathcal{H}_1(F) \rightarrow \mathfrak{h}_\lambda(H_0)$  is Hilbert-Schmidt, and therefore so is the operator  $\mathcal{E}_\lambda^\diamond(H_0): \mathfrak{h}_\lambda(H_0) \rightarrow \mathcal{H}_{-1}(F)$ .  $\square$

**5.1. Scattering operator.** We define the scattering operator as an operator

$$\mathbf{S}(H_1, H_0): \mathcal{S}(H_0, F) \rightarrow \mathcal{S}(H_0, F),$$

given by equality

$$(68) \quad \mathbf{S}(H_1, H_0) = \int_{\Lambda(H_0, F) \cap \Lambda(H_1, F)}^\oplus S(\lambda; H_1, H_0) d\lambda.$$



The usual definition of the scattering operator

$$\mathbf{S}(H_1, H_0) = W_+^*(H_1, H_0)W_-(H_1, H_0)$$

follows from definitions of the wave operators (65) and the scattering matrix (67). Obviously, for scattering operator we have analogues of properties of the scattering matrix given in Theorem 5.1, similar to those in [Az, Theorem 7.1.3]. In particular, the scattering operator  $\mathbf{S}(H_1, H_0)$  is unitary and commutes with  $H_0$ .

## 6. EXAMPLE

Recall that the class of potentials  $K_\nu$  on  $\mathbb{R}^\nu$  [S, p.453] is defined as follows: a real-valued measurable function  $V$  belongs to  $K_\nu$  if and only if

(a) if  $\nu \geq 3$

$$\lim_{\alpha \downarrow 0} \left[ \sup_x \int_{|x-y| \leq \alpha} |x-y|^{-(\nu-2)} |V(y)| d^\nu y \right] = 0,$$

(b) if  $\nu = 2$

$$\lim_{\alpha \downarrow 0} \left[ \sup_x \int_{|x-y| \leq \alpha} \ln(|x-y|^{-1}) |V(y)| d^2 y \right] = 0,$$

(c) if  $\nu = 1$

$$\sup_x \int_{|x-y| \leq 1} |V(y)| dy < \infty.$$

In particular,  $L^\infty(\mathbb{R}^\nu, \mathbb{R}) \subset K_\nu$ . A potential  $V$  belongs to  $K_\nu^{loc}$  if  $V\chi_R \in K_\nu$ , where  $\chi_R$  is the characteristic function of the ball  $\{x: |x| \leq R\}$ . The following theorem provides a large class of compatible pairs  $(H, F)$  among Schrödinger operators.

**Theorem 6.1.** [S, Theorem B.9.1] *Let  $\nu$  be any positive integer. Let  $H = -\Delta + V(x)$  be a Schrödinger operator on  $L^2(\mathbb{R}^\nu)$  with potential  $V$  satisfying  $V_- \in K_\nu$ ,  $V_+ \in K_\nu^{loc}$ . Let  $f$  be a bounded Borel function on the spectrum of  $H$  obeying  $|f(x)| \leq C(1+|x|)^{-\alpha}$  for some  $\alpha > \nu/4$ . If  $g \in L^2(\mathbb{R}^\nu)$ , then  $g(x)f(H)$  is Hilbert-Schmidt.*

It follows from this theorem that if  $F$  is multiplication by  $g(x) \in L^2(\mathbb{R}^\nu) \cap L^\infty(\mathbb{R}^\nu)$  where  $g(x) \neq 0$  for a.e.  $x \in \mathbb{R}$ , then for any dimension  $\nu$  the operator  $FE_\Delta^H$  is Hilbert-Schmidt. Hence, we have, in particular, a new proof of the following theorem as a corollary of part (i) of Theorem 4.7.

**Theorem 6.2.** *Let  $\nu$  be a positive integer and let  $H_0$  be self-adjoint extension of the differential operator  $-\Delta + V_0$  on  $\mathbb{R}^\nu$ , where  $V_0$  is a potential satisfying  $V_- \in K_\nu$ ,  $V_+ \in K_\nu^{loc}$ . For any self-adjoint operator  $H_1$  such that  $H_1 - H_0 \in L^\infty(\mathbb{R}^\nu) \cap L^1(\mathbb{R}^\nu)$ , the wave operators  $W_\pm(H_1, H_0)$  exist and complete.*

Other results of Sections 4 and 5 also apply to this class of Schrödinger operators.

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SCHOOL OF COMPUTER SCIENCE, ENGINEERING AND MATHEMATICS, FLINDERS UNIVERSITY, BEDFORD PARK, 5042, SA AUSTRALIA.

*E-mail address:* nurulla.azamov@flinders.edu.au